

A GRADIENT FLOW APPROACH TO THE BOLTZMANN EQUATION

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ABSTRACT. We show that the spatially homogeneous Boltzmann equation (with bounded collision kernel) evolves as the gradient flow of the entropy with respect to a suitable geometry on the space of probability measures. This geometry is given by a new notion of distance between probability measures, which takes the collision process into account. As first applications, we obtain a novel time-discrete approximation scheme for the homogeneous Boltzmann equation and a new simple proof for the convergence of Kac's random walk to the homogeneous Boltzmann equation.

1. INTRODUCTION

Since the pioneering work of Otto [15] it is well known that many diffusion equations can be cast as gradient flows of entropy functionals in the space of probability measures. The relevant geometry is induced by the L^2 Wasserstein distance. Otto's approach has been used as a powerful and versatile tool in the study of the trend to equilibrium, stability questions and construction of solutions. Prototypical examples that admit such a gradient flow formulation are Fokker–Planck and porous medium type equations, but many new instances have been exhibited by now. In each case—as a direct consequence of the gradient flow structure—the driving entropy functional is non-decreasing along the solution.

One of the most emblematic dissipative evolution equations is the Boltzmann equation modeling the evolution of a dilute gas under elastic collisions of the particles. Boltzmann's famous H-theorem asserts that the entropy is non-increasing along its solutions. Nevertheless, a gradient flow description of this equation has been out of reach so far.

In this article we remedy this fact and present a characterization of the spatially homogeneous Boltzmann equation as a gradient flow of the entropy. The crucial new insight is the identification of a novel geometry on the space of probability measures. It is induced by a distance between probability measures that takes the collision process between particles into account.

Let us describe the setting and the main results in more detail. We consider the spatially homogeneous Boltzmann equation

$$\partial_t f = Q(f, f) , \tag{1.1}$$

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where $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a probability density and Q denotes the Boltzmann collision operator given by

$$Q(f, f) = \int_{\mathbb{R}^d} \int_{S^{d-1}} [f' f'_* - f f_*] B(v - v_*, \omega) dv_* d\omega, \quad (1.2)$$

Here B is the collision kernel and v, v_* and v', v'_* denote the pre- and post-collisional velocities respectively which are related according to

$$v' = v - \langle v - v_*, \omega \rangle \omega, \quad v'_* = v_* + \langle v - v_*, \omega \rangle \omega, \quad \omega \in S^{d-1}. \quad (1.3)$$

We will often use the abbreviated notation

$$f = f(v), \quad f_* = f(v_*), \quad f' = f(v'), \quad f'_* = f(v'_*).$$

Boltzmann's H-theorem asserts that the entropy $\mathcal{H}(f) = \int f \log f$ is non increasing along solutions to the Boltzmann equation, more precisely, we have $\frac{d}{dt} \mathcal{H}(f_t) = -D(f_t) \leq 0$, where

$$D(f_t) = \frac{1}{4} \int \log \frac{f' f'_*}{f f_*} (f' f'_* - f f_*) B(v - v_*, \omega) d\omega dv_* dv. \quad (1.4)$$

1.1. Gradient flow structure. The gradient flow structure for the Boltzmann equation rests on a novel geometry on the space of probability measures. We will define a distance \mathcal{W}_B between probabilities μ and ν by setting

$$\mathcal{W}_B(\mu, \nu)^2 = \inf \left\{ \frac{1}{4} \int_0^1 \int |\bar{\nabla} \psi_t|^2 \Lambda(f_t) B(v - v_*, \omega) d\omega dv_* dv dt \right\}, \quad (1.5)$$

where the infimum runs over all curves of probability measures $t \mapsto f_t dv$ connecting μ and ν and all functions $\psi : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ related via

$$\partial_t f_t(v) + \frac{1}{4} \int \bar{\nabla} \psi_t \Lambda(f_t) B(v - v_*, \omega) d\omega dv_* = 0. \quad (1.6)$$

Here, we have set $\bar{\nabla} \varphi = \varphi' + \varphi'_* - \varphi - \varphi_*$ and $\Lambda(f)$ is shorthand for $\Lambda(f f_*, f' f'_*)$, where $\Lambda(s, t) = (s - t)/(\log s - \log t)$ denotes the logarithmic mean.

Note that the definition of \mathcal{W}_B resembles the dynamic formulation of the L^2 -Wasserstein distance, known as the Benamou–Brenier formula [3]. Here, the *collision rate equation* (1.6) takes over the role of the usual continuity equation. Roughly speaking, the potential ψ governs the evolution of the density f by prescribing the rate at which collisions happen between the particles.

To motivate this approach, note that in analogy to the Otto calculus for the Wasserstein distance, the function $\bar{\nabla} \psi$ should be regarded as the tangent vector to the curve $(f_t)_t$. The formal metric tensor on the space of probability measures which gives rise to the distance \mathcal{W}_B is then given by

$$\langle \bar{\nabla} \psi, \bar{\nabla} \varphi \rangle_f = \frac{1}{4} \int \bar{\nabla} \psi \bar{\nabla} \varphi \Lambda(f) B(v - v_*, \omega) d\omega dv_* dv.$$

If f_t is a solution to the Boltzmann equation, it satisfies (1.6) with $\psi_t = \log f_t$ and we find formally the gradient flow relation

$$\frac{d}{dt} \mathcal{H}(f_t) = -|\bar{\nabla} \log f_t|_{f_t}^2 = -|\text{grad } \mathcal{H}(f_t)|^2.$$

This gradient flow structure bears some analogy with work of Maas [11] and Mielke [12], where gradient flow structures for finite Markov chains and reaction-diffusion equations have been found. Formally, the homogeneous Boltzmann equation could be seen as a binary reaction equation with a continuum of species indexed by the velocity.

We will first construct the distance \mathcal{W}_B by a suitable relaxation of the minimization problem (1.5), (1.6) to a measure-valued framework. Here, we restrict ourselves to considering collision kernels B which are integrable in the angular variable and have a certain continuity in the velocity variable, see Assumption 2.1 for the precise conditions we require. However, we believe that the approach is robust and flexible enough to deal also with singular kernels. Our first result is the following (see Theorem 3.18 below).

Theorem 1.1. *Let B satisfy Assumption 2.1. Then \mathcal{W}_B defines an extended (i.e. potentially taking the value $+\infty$), separable and complete distance on the set $\mathcal{P}_*(\mathbb{R}^d)$ of probability measures with zero mean and unit variance. Each pair μ, ν with $\mathcal{W}_B(\mu, \nu) < \infty$ can be joined by a geodesic.*

Under the additional assumption that the collision kernel B is bounded from above and below, we rigorously characterize the spatially homogeneous Boltzmann equation (1.1) as the gradient flow of the entropy in $\mathcal{P}_*(\mathbb{R}^d)$ w.r.t. the distance \mathcal{W}_B . We obtain the following variational characterization.

Theorem 1.2. *Let B satisfy Assumption 4.1. Then for any curve $(f_t)_{t \geq 0}$ curve of probability densities in $\mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(f_0) < \infty$ we have that*

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2} \int_0^T D(f_t) + |\dot{f}|^2(t) dt \geq 0 \quad \forall T \geq 0,$$

where $|\dot{f}|(t)$ denotes the metric speed w.r.t. \mathcal{W}_B (see (2.11) for the definition). We have $J_T(f) = 0$ for all T if and only if $(f_t)_t$ is the solution to the homogeneous Boltzmann equation starting from f_0 .

In this sense, the Boltzmann equation is a steepest descent decreasing the entropy “as fast as possible”. To motivate this result, note that for a smooth function $E : \mathbb{R}^n \rightarrow \mathbb{R}$ and any smooth curve $x : [0, \infty) \rightarrow \mathbb{R}^n$ we have by the Cauchy-Schwartz and Young inequalities

$$E(x_T) - E(x_0) = \int_0^T \nabla E(x_t) \dot{x}_t dt \geq -\frac{1}{2} \int_0^T |\nabla E|^2(x_t) + |\dot{x}_t|^2 dt. \quad (1.7)$$

Moreover, equality holds if and only if x_t is a gradient flow curve of E , i.e. $\dot{x}_t = -\nabla E(x_t)$. In a metric space such as $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$ the gradient flow of E can be defined by requiring the equality (1.7) with $|\dot{x}_t|$ replaced by the metric speed of the curve and $|\nabla E|$ replaced by an upper gradient, i.e. a function such that \geq in (1.7) holds for any curve. We refer to Section 2.4 and [1] for more details on gradient flows in metric spaces. Precisely, we will show in the terminology of [1] that \sqrt{D} is a strong upper gradient of \mathcal{H} in $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$ and that the solutions of the Boltzmann equation are precisely the curves of maximal slope, see Proposition 4.2 and Theorem 4.4 below.

The assumption that the collision kernel is bounded above and below is imposed for technical reasons to have regularization techniques involving

the Ornstein–Uhlenbeck semigroup at our disposal. We believe that this gradient flow structure should be present for much more general kernels.

1.2. Variational approximation scheme. As a first application, we obtain a novel time-discrete variational approximation scheme for the Boltzmann equation. In fact, we show that the so-called minimizing movement scheme, i.e. the implicit Euler scheme for the gradient flow equation, converges to the solution to (1.1) (see Theorem 5.1 below). This is reminiscent of the results by Jordan–Kinderlehrer–Otto [9] for the heat equation.

Given a time step $\tau > 0$ and an initial datum $f_0 \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(f_0) < \infty$ let $(\mu_n^\tau)_n$ be defined recursively via

$$f_0^\tau = f_0, \quad f_n^\tau \in \operatorname{argmin}_g \left[\mathcal{H}(g) + \frac{1}{2\tau} \mathcal{W}_B(g, f_{n-1}^\tau)^2 \right].$$

Define a piece-wise constant interpolation $(\bar{f}_t^\tau)_{t \geq 0}$ via

$$\bar{f}_0^\tau = f_0, \quad \bar{f}_t^\tau = f_n^\tau \text{ if } t \in ((n-1)\tau, n\tau].$$

Theorem 1.3. *Let B satisfy Assumption 4.1. As τ goes to zero, we have that \bar{f}_t^τ converges weakly to f_t for all t , where $(f_t)_t$ is the solution to the Boltzmann equation with initial datum f_0 .*

1.3. Consistency for Kac’s random walk. As a second application we use the gradient flow structure to give a new and simple proof of the convergence of Kac’s random walk to the solution of the spatially homogeneous Boltzmann equation. Kac introduced his random walk in the seminal work [10] as a probabilistic model for N -colliding particles. It is a continuous time Markov chain on the set \mathcal{X}_N of N velocities with fixed momentum and energy,

$$\mathcal{X}_N := \left\{ (v_1, \dots, v_N) \in \mathbb{R}^{dN} \mid \sum_{i=1}^N v_i = 0, \sum_{i=1}^N |v_i|^2 = N \right\}.$$

In each step, two uniformly chosen particles i, j collide, i.e. \mathbf{v} is updated to $R_{ij}^\omega \mathbf{v} = (v_1, \dots, v'_i, \dots, v'_j, \dots, v_N)$ where $v'_i = v_i - \langle v_i - v_j, \omega \rangle \omega$ and $v'_j = v_j + \langle v_i - v_j, \omega \rangle \omega$ with a uniformly chosen collision parameter $\omega \in S^{d-1}$. The rate is chosen such that on average N collision occur per unit of time. More precisely, the generator of the Markov chain is given by

$$Af(\mathbf{v}) = \frac{1}{N} \int_{S^{d-1}} \sum_{i,j=1}^N [f(R_{ij}^\omega \mathbf{v}) - f(\mathbf{v})] B(v_i - v_j, \omega) d\omega, \quad (1.8)$$

The Markov chain is reversible with respect to the Hausdorff measure π_N on \mathcal{X}_N . If we denote by μ_t^N the law of the Markov chain starting from μ_0^N , then its density f_t^N w.r.t. π_N satisfies Kac’s master equation $\partial_t f_t^N = Af_t^N$. A natural way to study the convergence of Kac’s random walk to the Boltzmann equation is via its empirical measures $L_N(\mathbf{v}) = \frac{1}{N} \sum_{i=1}^N \delta_{v_i} \in \mathcal{P}_*(\mathbb{R}^d)$. We will show the following:

Theorem 1.4. *Let B satisfy Assumption 4.1. For each N let $(\mu_t^N)_{t \geq 0}$ be a the law of Kac’s random walk starting from μ_0^N and denote by $c_t^N := (L_N)_\# \mu_t^N \in \mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ be the law of its empirical measures. Assume that*

μ_0^N is well-prepared for some $\nu_0 = f_0 \mathcal{L} \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\nu_0) < \infty$ in the sense that in the limit $N \rightarrow \infty$

$$c_0^N \rightharpoonup \delta_{\nu_0}, \quad \frac{1}{N} \mathcal{H}_N(\mu_0^N) \rightarrow \mathcal{H}(\nu_0) - \mathcal{H}_0.$$

Then, for all $t > 0$, c_t^N converges weakly to δ_{ν_t} , where $\nu_t = f_t \mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0 . Moreover, we have that

$$\frac{1}{N} \mathcal{H}_N(\mu_t^N) \rightarrow \mathcal{H}(\nu_t) - \mathcal{H}_0. \quad (1.9)$$

Here \mathcal{H}_N denotes the relative entropy w.r.t π_N and $\mathcal{H}_0 := \inf\{\mathcal{H}(\nu) : \nu \in \mathcal{P}_*\}$. In fact we have $\mathcal{H}_0 = \mathcal{H}(M)$, where M is the standard Gaussian density. Note that the well-preparedness assumption is satisfied for instance if the initial velocities are independent, i.e. $\mu_0^N = \nu_0^{\otimes N}$. An important feature of Kac's model is the *propagation of chaos*: if the initial distribution of velocities is asymptotically independent as $N \rightarrow \infty$ then the same holds for all times. One way of making this precise is the convergence (1.9), which is usually called entropic propagation of chaos. This is motivated by the fact that for a true product measure we have $\mathcal{H}(\nu^{\otimes N}) = N \cdot \mathcal{H}(\nu)$.

We hasten to point out that Theorem 1.4 is not new but rather well-known. Kac proved an analogue for a simplified model with one-dimensional velocities in [10]. The proof of convergence to the homogeneous Boltzmann equation for the model considered here goes back to Sznitman [17]. Much finer quantitative convergence results in Wasserstein distance were obtained by Mischler–Mouhot [13] and Norris [14]. Also the convergence is known for a variety of collision kernels.

The contribution we make here is to give a new and simple proof of convergence based on the gradient flow structure. We will use the stability of gradient flows following the approach of Sandier–Serfaty [16]. It turns out that Kac's random walk is the gradient flow of the entropy \mathcal{H}_N in $\mathcal{P}(\mathcal{X}_N)$ equipped with a suitable transport distance \mathcal{W}_N , as we shall make precise in Section 6.1. In particular, the energy dissipation equality

$$\mathcal{H}_N(\mu_t^N) - \mathcal{H}_N(\mu_0^N) = \frac{1}{2} \int_0^t D_N(\mu_r^N) + |\dot{\mu}^N|_N^2(r) dr \quad (1.10)$$

holds, where D_N is the dissipation of \mathcal{H}_N along the master equation and $|\dot{\mu}^N|_N$ is the metric speed w.r.t. \mathcal{W}_N . This is based on results for general Markov chains and jump processes in [11, 12, 8]. To obtain the desired convergence to the Boltzmann equation it is sufficient together with some compactness to prove convergence (in fact only \liminf estimates) for the constituent elements of the gradient flow structure, the entropy, dissipation and the metric speed, which allow to pass to the limit in (1.10).

Organization. In Section 2 we collect necessary preliminaries, in particular we recall regularizing properties of the Ornstein–Uhlenbeck semigroup in the context of the Boltzmann equation and the basic framework of gradient flows in metric spaces. In Section 3 we present the construction of the distance \mathcal{W}_B and establish various properties. The characterization of the Boltzmann equation as entropic gradient flow is obtained in Section 4. Applications

are considered in Section 5 to the variational approximation scheme and in Section 6 to the convergence of Kac random walk to the Boltzmann equation.

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2. PRELIMINARIES

2.1. Homogeneous Boltzmann equation, entropy and dissipation.

Throughout the paper we make the following assumption on the collision kernel $B : \mathbb{R}^d \times S^{d-1} \rightarrow \mathbb{R}_+$ unless stated otherwise.

Assumption 2.1. *There exists a constant C_B such that*

$$\int_{S^{d-1}} B(k, \omega) d\omega \leq C_B \quad \forall k \in \mathbb{R}^d. \quad (2.1)$$

Moreover, for any function $g \in C_c^\infty(\mathbb{R}^{2d} \times S^{d-1})$ the map

$$(v, v_*) \mapsto \int_{S^{d-1}} g(v, v_*, \omega) B(v - v_*, \omega) d\omega$$

is continuous and bounded.

We recall well known existence and uniqueness results for the homogeneous Boltzmann equation in this setting going back to [2].

Theorem 2.2. *Let $f_0 : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be such that*

$$\int_{\mathbb{R}^d} (1 + |v|^2) f_0(v) dv < \infty, \quad \int_{\mathbb{R}^d} f_0(v) \log f_0(v) dv < \infty.$$

Then there exists a unique classical solution $(f_t)_{t \geq 0}$ to the homogeneous Boltzmann equation (1.1). It conserves mass, momentum and energy, i.e.

$$\int (1, v, |v|^2) f_t(v) dv = \int (1, v, |v|^2) f_0(v) dv \quad \forall t \geq 0.$$

We denote by $\mathcal{P}(\mathbb{R}^d)$ the space of Borel probability measures on \mathbb{R}^d . Moreover, we denote

$$\mathcal{P}_*(\mathbb{R}^d) := \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) \mid \int v d\mu(v) = 0, \int |v|^2 d\mu(v) = 1 \right\}. \quad (2.2)$$

If μ has a density f w.r.t. the Lebesgue measure \mathcal{L} on \mathbb{R}^d we write also $f \in \mathcal{P}_*(\mathbb{R}^d)$. Theorem 2.2 asserts in particular, that $f_t \in \mathcal{P}_*(\mathbb{R}^d)$ for all t provided $f_0 \in \mathcal{P}_*(\mathbb{R}^d)$.

We denote by $\mathcal{H}(\mu)$ the *Boltzmann–Shannon entropy* defined for $\mu \in \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{H}(\mu) = \int f(v) \log f(v) dv,$$

provided $\mu = f\mathcal{L}$ is absolutely continuous and $\max(f \log f, 0)$ is integrable, otherwise $\mathcal{H}(\mu) = +\infty$.

Let M denote the standard Maxwellian distribution in \mathbb{R}^d , i.e.

$$M(v) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|v|^2}{2}\right).$$

The relative entropy w.r.t. M of a probability measure $\mu = gM\mathcal{L}$ is defined by

$$\mathcal{H}(\mu|M) = \int g(v) \log g(v) M(v) dv. \quad (2.3)$$

Note that $\mathcal{H}(\mu) = \mathcal{H}(\mu|M) + \mathcal{H}(M)$. By Jensen's inequality we have $\mathcal{H}(\cdot|M) \geq 0$. Thus for any $\mu \in \mathcal{P}_*(\mathbb{R}^d)$ we obtain the bound

$$\mathcal{H}(\mu) = \mathcal{H}(\mu|M) - \frac{1}{2} \int |v|^2 \mu(dv) - \frac{N}{2} \log(2\pi) \geq -\frac{1}{2} - \frac{N}{2} \log(2\pi). \quad (2.4)$$

The first identity also shows that \mathcal{H} is lower semicontinuous on $\mathcal{P}_*(\mathbb{R}^d)$ w.r.t. weak convergence, i.e. in duality with bounded continuous functions. This follows from the corresponding property of $\mathcal{H}(\cdot|M)$ and the second moment.

Boltzmann's H-theorem asserts that the entropy is non-increasing along the solution f_t of the Boltzmann equation. More precisely, in the setting of Theorem 2.2 we have (see [2]):

$$\frac{d}{dt} \mathcal{H}(f_t) = -D(f_t) \leq 0,$$

where

$$D(f) := \int_{\mathbb{R}^{2N}} \int_{S^{d-1}} \log \frac{f' f'_*}{f f_*} [f' f'_* - f f_*] B(v - v_*, \omega) dv dv_* d\omega. \quad (2.5)$$

The quantity $D(f)$ is called the *entropy dissipation*. More generally, we define the entropy dissipation $D(\mu)$ for a probability measure μ by setting $D(\mu) = D(f)$, provided $\mu = f\mathcal{L}$ is absolutely continuous and $+\infty$ otherwise.

2.2. Ornstein–Uhlenbeck regularization. We recall that the (adjoint) Ornstein–Uhlenbeck semigroup $(S_t)_{t \geq 0}$ can be defined as a rescaled convolution with the standard Maxwellian distribution M . For $f \in L^1(\mathbb{R}^d)$ and $t \geq 0$ we have

$$S_t f = f_{e^{-2t}} * M_{1-e^{-2t}},$$

with the notation $g_\lambda(v) = \frac{1}{\lambda^{d/2}} g\left(\frac{v}{\lambda}\right)$. Recall that $f_t := S_t f$ is the solution to the Fokker–Planck equation $\partial_t f = \nabla \cdot (\nabla f + f v)$, $f_0 = f$. We note that for any $f \in L^1$, $S_t f$ is C^∞ with the Gaussian bound

$$|\log S_t f|(v) \leq C_t(1 + |v|^2), \quad (2.6)$$

for a suitable constant C_t , see for instance [5].

For fixed $\omega \in S^{d-1}$ we will denote by T_ω the transformation $(v, v_*) \mapsto (v', v'_*)$ with v', v'_* given by (1.3). Note that T_ω is involutive and has unit Jacobian determinant. We will set

$$X = (v, v_*), \quad X' = (v', v'_*) = T_\omega X.$$

By abuse of notation we denote the standard Maxwellian distribution and the Ornstein–Uhlenbeck semigroup in \mathbb{R}^{2d} again by M and S_t . Note that $M(X) := M(v)M(v_*)$. For a function $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ we will set

$$T_\omega F(X) := F(T_\omega X) .$$

It is readily checked that the operations of scaling, convolution and the semigroup S_t behave well under tensorization. More precisely, if for a function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we set $F = f \otimes f$, i.e. $F(X) = ff_*$, then we have

$$F_\lambda = f_\lambda \otimes f_\lambda , F * M_\delta = (f * M_\delta) \otimes (f * M_\delta) , S_t F = (S_t f) \otimes (S_t f) .$$

The following commutation relation with the pre-post-collision change of variables will be crucial in the sequel. It can be found in [18, Prop. 4]. For the reader's convenience we will give the short proof.

Lemma 2.3. *Let $F : \mathbb{R}^{2d} \rightarrow \mathbb{R}$. Then, we have that for each $\omega \in S^{d-1}$ and any $\lambda, \delta > 0$:*

$$(T_\omega F)_\lambda = T_\omega(F_\lambda) , \quad (T_\omega F) * M_\delta = T_\omega(F * M_\delta) . \quad (2.7)$$

In particular, for each $t \geq 0$ we have that:

$$S_t(T_\omega F) = T_\omega(S_t F) . \quad (2.8)$$

If $F = ff_$ we have for short $S_t(f'f'_*) = (S_t f)'(S_t f)'_*$.*

Proof. Since $S_t F$ can be written as a composition of scaling of F and a convolution with (a scaling of) M , the commutation (2.8) is a direct consequence of (2.7). Commutation of T_ω with the scaling operation is readily checked. It remains to check commutation with convolution. First note that $M_\delta(T_\omega X) = M_\delta(X)$, since the relation between pre- and post-collisional velocities is such that $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$. Using also the fact that T_ω is involutive with unit determinant, we find

$$\begin{aligned} (T_\omega F) * M_\delta(X) &= \int_{\mathbb{R}^{2d}} F(T_\omega Y) M_\delta(X - Y) dY \\ &= \int_{\mathbb{R}^{2d}} F(Y) M_\delta(X - T_\omega^{-1} Y) dY \\ &= \int_{\mathbb{R}^{2d}} F(Y) M_\delta(T_\omega X - Y) dY \\ &= (F * M_\delta)(T_\omega X) . \end{aligned}$$

□

2.3. Truncated moment estimate. We will make use of the following estimate which is in the spirit of the Povzner inequalities. Given $R > 0$ we write

$$\varphi_R(v) = |v| \wedge R .$$

Lemma 2.4. *There is a constant C such that for all $v, v_* \in \mathbb{R}^d$ and $\sigma \in S^{d-1}$ and $R > 0$ we have*

$$\left| \varphi_R(v')^2 + \varphi_R(v'_*)^2 - \varphi_R(v)^2 - \varphi_R(v_*)^2 \right| \leq C \left[|v| |v_*| + |v'| |v'_*| \right] , \quad (2.9)$$

where as usual v', v'_* are given by (1.3).

Proof. We divide the proof into several cases.

Case 1: None of the four points v, v_*, v', v'_* lies in $B_R(0)$. Here the LHS of (2.9) is 0.

Case 2: Exactly one point lies in $B_R(0)$. Without restriction let $v \in B_R(0)$. Then we have

$$LHS = R^2 - |v|^2 \leq |v'| |v'_*| ,$$

since $|v'|, |v'_*| \geq R$.

Case 3: Exactly two points lie in $B_R(0)$. Then without restriction we have either a) $v, v_* \in B_R(0)$ or b) $v, v' \in B_R(0)$. In case a) we have $LHS = 2R^2 - |v|^2 - |v_*|^2$ and we argue as in case 2. In case b) we have

$$LHS = \left| |v'|^2 - |v|^2 \right| \leq |v'| |v'_*| + |v| |v_*| ,$$

since $|v_*| \geq |v|$ and $|v'_*| \geq |v'|$.

Case 4: Exactly three points lie in $B_R(0)$. Without restriction let $v, v_*, v' \in B_R(0)$. We have

$$LHS = \left| |v'|^2 + R^2 - |v|^2 - |v_*|^2 \right| .$$

We make a further distinction of cases.

- a) $v' \notin B_{R/2}(0)$ or $v, v_* \notin B_{R/2}(0)$. Then $|v| |v_*| + |v'| |v'_*| \geq R^2/4$ while $LHS \leq 2R^2$.
- b) $v', v \in B_{R/2}(0)$ and $v_* \notin B_{R/2}(0)$. Write $v' = v + n$, $v'_* = v_* - n$ with $n = v - v' = -\omega \langle \omega, v - v_* \rangle$. By energy conservation we have

$$\begin{aligned} LHS &= |v'_*|^2 - R^2 \leq |v_*|^2 + 2|v_*| |n| + |n|^2 - R^2 \\ &\leq 2|v_*| (|v| + |v'|) + 2|v|^2 + 2|v'|^2 \\ &\leq 4(|v| |v_*| + |v'| |v'_*|) , \end{aligned}$$

since $|v| \leq |v_*|$ and $|v'| \leq |v'_*|$.

- c) $v', v_* \in B_{R/2}(0)$ and $v \notin B_{R/2}(0)$. Here we argue as in b) by writing $v' = v_* + m$, $v'_* = v - m$ for suitable m .

These are all sub-cases of 4, since $v, v_*, v' \in B_{R/2}(0)$ would contradict $v'_* \notin B_R(0)$.

Case 5: All four points are in $B_R(0)$. Here again the LHS is 0 by energy conservation.

□

2.4. Gradient flows in metric spaces. In this section we briefly recall the basic theory of gradient flow in metric spaces. For a detailed account we refer the reader to [1].

Let (X, d) be a complete and separable metric space and let $E : X \rightarrow (-\infty, \infty]$ be a function with proper domain, i.e. the set $D(E) := \{x : E(x) < \infty\}$ is non-empty.

A curve $(x_t)_{t \in (a,b)}$ in (X, d) is called p -absolutely for $p \geq 1$ if there exists $m \in L^p((a, b))$ such that

$$d(x_s, x_t) \leq \int_s^t m(r) dr \quad \forall a \leq s \leq t \leq b. \quad (2.10)$$

In this case we write $x \in AC^p((a, b); (X, d))$. For $p = 1$ we simply drop p in the notation. Similarly, one defines locally p -absolutely continuous curves. For a locally absolutely continuous curve the metric derivative defined by

$$|\dot{x}|(t) := \lim_{h \rightarrow 0} \frac{d(x_{t+h}, x_t)}{|h|} \quad (2.11)$$

exists for a.e. t and is the minimal m in (2.10), see [1, Thm.1.1.2].

The following notion plays the role of the modulus of the gradient in a metric setting.

Definition 2.5 (Strong upper gradient). *A function $g : X \rightarrow [0, \infty]$ is called a strong upper gradient of E if for any $x \in AC((a, b); (X, d))$ the function $g \circ x$ is Borel and*

$$|E(x_s) - E(x_t)| \leq \int_s^t g(x_r) |\dot{x}|(r) dr \quad \forall a \leq s \leq t \leq b.$$

Note that by the definition of strong upper gradient, and Young's inequality $ab \leq \frac{1}{2}(a^2 + b^2)$, we have that for all $s \leq t$:

$$E(x_t) - E(x_s) + \frac{1}{2} \int_s^t g(x_r)^2 + |\dot{x}|^2(r) dr \geq 0.$$

Definition 2.6 (Curve of maximal slope). *A locally 2-absolutely continuous curve $(x_t)_{t \in (0, \infty)}$ is called a curve of maximal slope of E w.r.t. its strong upper gradient g if and only if $t \mapsto E(x_t)$ is non-increasing and*

$$E(x_t) - E(x_s) + \frac{1}{2} \int_s^t g(x_r)^2 + |\dot{x}|^2(r) dr \leq 0 \quad \forall 0 < s \leq t. \quad (2.12)$$

We say that a curve of maximal slope starts from $x_0 \in X$ if and only if $\lim_{t \searrow 0} x_t = x_0$.

Equivalently, we can require equality in (2.12). If a strong upper gradient g of E is fixed we also call a curve of maximal slope of E (relative to g) a *gradient flow curve*.

Finally, we define the (descending) metric slope of E as the function $|\partial E| : D(E) \rightarrow [0, \infty]$ given by

$$|\partial E|(x) = \limsup_{y \rightarrow x} \frac{\max\{E(x) - E(y), 0\}}{d(x, y)}. \quad (2.13)$$

The metric slope is in general only a weak upper gradient E , see [1, Thm. 1.2.5]. In our application to the homogeneous Boltzmann equation, we will show that the square root of the dissipation D provides a strong upper gradient for the entropy \mathcal{H} .

3. THE COLLISION DISTANCE

In this section, we present a new type of distance between probability measures on \mathbb{R}^d . It will be constructed by minimizing an action functional over curves in the space of probability measures.

3.1. The action functional. First we need to introduce some notation. We let

$$G = \mathbb{R}^d \times \mathbb{R}^d \times S^{d-1}$$

and denote by $\mathcal{M}(G)$ the space of signed Radon measures on G equipped with the weak* topology in duality with continuous functions with compact support in G . Recall that $\mathcal{P}(\mathbb{R}^d)$ denotes the space of Borel probability measures on \mathbb{R}^d equipped with the topology of weak convergence in duality with bounded continuous functions.

We introduce a function $\Lambda : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$\Lambda(s, t) = \int_0^1 s^\alpha t^{1-\alpha} d\alpha = \frac{s - t}{\log s - \log t}, \quad (3.1)$$

the latter expression being valid for positive $s \neq t$. This is called the logarithmic mean of s and t . It will be useful to note that Λ is concave and positively homogeneous, i.e. $\Lambda(\alpha s, \alpha t) = \alpha \Lambda(s, t)$ for all $\alpha \geq 0$. Moreover it is easy to check that

$$\Lambda(s, t) \leq \frac{s + t}{2} \quad \forall s, t \geq 0. \quad (3.2)$$

Given a function $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ we will often write

$$\Lambda(f)(v, v_*, \omega) = \Lambda(f f_*, f' f'_*).$$

We can now define a function $\alpha : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$ by setting

$$\alpha(u, s, t) := \begin{cases} \frac{u^2}{4\Lambda(s, t)}, & \Lambda(s, t) \neq 0, \\ 0, & \Lambda(s, t) = 0 \text{ and } u = 0, \\ +\infty, & \theta(s, t) = 0 \text{ and } u \neq 0. \end{cases} \quad (3.3)$$

The following observation will be useful.

Lemma 3.1. *The function α is lower semicontinuous, convex and positively homogeneous, i.e.*

$$\alpha(ru, rs, rt) = r\alpha(u, s, t) \quad \forall u \in \mathbb{R}, s, t \geq 0, r \geq 0.$$

Proof. This is easily checked using homogeneity and concavity of Λ and the convexity of the function $(u, y) \mapsto \frac{u^2}{y}$ on $\mathbb{R} \times (0, \infty)$. \square

We will now define an action functional on pairs of measures (μ, \mathcal{U}) where $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U} \in \mathcal{M}(G)$. To μ we associate two measures in $\mathcal{M}(G)$ by setting:

$$\mu^1(dv, dv_*, d\omega) := B(v - v_*, \omega) \mu(dv) \mu(dv_*) d\omega, \quad \mu^2 := T_{\#} \mu^1, \quad (3.4)$$

where T is the change of variables $(v, v_*, \omega) \mapsto (T_\omega(v, v_*), \omega)$ between pre- and post-collisional variables defined in (1.3). We can always choose a measure $\tau \in \mathcal{M}(G)$ such that $\mu^i = f^i \tau$, $i = 1, 2$ and $\mathcal{U} = U \tau$ are all absolutely continuous with respect to τ , for instance by taking the sum of the total

variations $\tau := |\mu^1| + |\mu^2| + |\mathcal{U}|$. We can then define the *action functional* by

$$\mathcal{A}(\mu, \mathcal{U}) := \int \alpha(U, f^1, f^2) d\tau .$$

Note that this definition is independent of the choice of τ since α is positively homogeneous. Another way to write the action functional is

$$\mathcal{A}(\mu, \mathcal{U}) = \int \alpha \left(\frac{d\lambda_1}{d|\lambda|}, \frac{d\lambda_2}{d|\lambda|}, \frac{d\lambda_3}{d|\lambda|} \right) d|\lambda| ,$$

where λ is the vector valued measure given by $\lambda = (\mathcal{U}, \mu^1, \mu^2)$.

If the measure μ is absolutely continuous w.r.t. the Lebesgue measure \mathcal{L} on \mathbb{R}^d , the next lemma shows that the action takes a more intuitive form. For this we denote by $\mathcal{B} \in \mathcal{M}(G)$ the measure given by

$$\mathcal{B}(dv, dv_*, d\omega) = B(v - v_*, \omega) dv dv_* d\omega .$$

Lemma 3.2. *Let $\mu \in \mathcal{P}(\mathbb{R}^d)$ be absolutely continuous w.r.t. \mathcal{L} with density f . Further, let $\mathcal{U} \in \mathcal{M}(G)$ be such that $\mathcal{A}(\mu, \mathcal{U}) < \infty$. Then there exist a function $U : G \rightarrow \mathbb{R}$ such that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ and we have*

$$\mathcal{A}(\mu, \mathcal{U}) = \frac{1}{4} \int |U(v, v_*, \omega)|^2 \Lambda(f)\mathcal{B}(v - v_*, \omega) dv dv_* d\omega . \quad (3.5)$$

Proof. Choose $\lambda \in \mathcal{M}(G)$ such that $\mathcal{B} = h\lambda$ and $\mathcal{U} = \tilde{U}\lambda$ are both absolutely continuous w.r.t. λ . Note that $\mu^i = \rho^i\mathcal{B}$, $i = 1, 2$ with $\rho^1(v, v_*, \omega) = f(v)f(v_*)$ and $\rho^2(v, v_*, \omega) = f(v')f(v'_*)$. Further, we denote by $\tilde{\rho}^i$ the density of μ^i w.r.t. λ . Now by definition,

$$\mathcal{A}(\mu, \mathcal{U}) = \int \alpha(\tilde{U}, \tilde{\rho}^1, \tilde{\rho}^2) d\lambda < \infty . \quad (3.6)$$

Let $A \subset G$ be such that $\int_A \theta(\rho^1, \rho^2) d\mathcal{B} = 0$. From the homogeneity of Λ we conclude

$$0 = \int_A \Lambda(\rho^1, \rho^2) d\mathcal{B} = \int_A \Lambda(\tilde{\rho}^1, \tilde{\rho}^2) d\lambda ,$$

i.e. $\Lambda(\tilde{\rho}^1, \tilde{\rho}^2) = 0$ λ -a.e. on A . Now the finiteness of the integral in (3.6) implies that $\tilde{U} = 0$ λ -a.e. on A . In other words $\mathcal{U}(A) = 0$ and hence \mathcal{U} is absolutely continuous w.r.t. the measure $\Lambda(f)\mathcal{B}$. Formula (3.5) now follows immediately from the homogeneity of α . \square

In view of the previous lemma, given a pair of functions $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ and $U : G \rightarrow \mathbb{R}$ we will define their action via $\mathcal{A}(f, U) := \mathcal{A}(\mu, \mathcal{U})$ with $\mu = f\mathcal{L}$ and $\mathcal{U} = \Lambda(f)\tilde{\mathcal{B}}$.

Next, we establish lower semicontinuity of the action w.r.t. convergence of μ and \mathcal{U} .

Lemma 3.3 (Lower semicontinuity of the action). *\mathcal{A} is lower semicontinuous w.r.t. weak convergence of measures. More precisely, assume that $\mu_n \rightharpoonup \mu$ weakly in $\mathcal{P}(\mathbb{R}^d)$ and $\mathcal{U}_n \rightharpoonup^* \mathcal{U}$ weakly* in $\mathcal{M}(G)$. Then*

$$\mathcal{A}(\mu, \mathcal{U}) \leq \liminf_n \mathcal{A}(\mu_n, \mathcal{U}_n) .$$

Proof. Note that by the Assumption 2.1 on the collision kernel B , the weak convergence of μ_n to μ implies the weak* convergence of μ_n^i to μ^i in $\mathcal{M}(G)$ for $i = 1, 2$. Now the claim follows immediately from a general result on integral functionals, Proposition 3.4. \square

Proposition 3.4 ([4, Thm. 3.4.3]). *Let Ω be a locally compact Polish space and let $f : \Omega \times \mathbb{R}^d \rightarrow [0, +\infty]$ be a lower semicontinuous function such that $f(\omega, \cdot)$ is convex and positively 1-homogeneous for every $\omega \in \Omega$. Then the functional*

$$F(\lambda) = \int_{\Omega} f\left(\omega, \frac{d\lambda}{d|\lambda|}(\omega)\right) |\lambda| (d\omega)$$

is sequentially weak lower semicontinuous on the space of vector valued signed Radon measures $\mathcal{M}(\Omega, \mathbb{R}^d)$.*

The next estimate will be crucial for establishing compactness of families of curves with bounded action in Section 3.2.

Lemma 3.5. *For any measurable function $\Psi : \mathbb{R}^{2d} \times S^{d-1} \rightarrow \mathbb{R}_+$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\mathcal{U} \in \mathcal{M}(G)$ we have:*

$$\int \Psi d|\mathcal{U}| \leq 2\mathcal{A}(\mu, \mathcal{U})^{\frac{1}{2}} \left(\frac{1}{2} \int (\Psi^2 + \Psi^2 \circ T)(v, v_*, \omega) B(v - v_*, \omega) d\omega d\mu(v) d\mu(v_*) \right)^{\frac{1}{2}}.$$

Proof. Let us write $\mu^i = \rho^i \tau$, $\mathcal{U} = U\tau$ for a suitable measure τ . We can assume that $\mathcal{A}(\mu, \mathcal{U}) < \infty$ as otherwise there is nothing to prove. Hence, the set $A = \{(v, v_*, \omega) \mid \alpha(U, \rho^1, \rho^2) = \infty\}$ has zero measure with respect to τ . We can now estimate:

$$\begin{aligned} \int \Psi d|\mathcal{U}| &\leq \int \Psi |U| d\tau = 2 \int_{A^c} \Psi \sqrt{\Lambda(\rho^1, \rho^2)} \sqrt{\alpha(w, \rho^1, \rho^2)} d\tau \\ &\leq 2 \left(\int \alpha(w, \rho^1, \rho^2) d\tau \right)^{\frac{1}{2}} \left(\int \Psi^2 \Lambda(\rho^1, \rho^2) d\tau \right)^{\frac{1}{2}} \\ &\leq 2\mathcal{A}(\mu, \mathcal{U})^{\frac{1}{2}} \left(\frac{1}{2} \int (\Psi^2 + \Psi^2 \circ T)(v, v_*, \omega) B(v - v_*, \omega) d\omega d\mu(v) d\mu(v_*) \right)^{\frac{1}{2}}. \end{aligned}$$

The last inequality follows from the estimate (3.2):

$$\begin{aligned} \int \Psi^2 \Lambda(\rho^1, \rho^2) d\tau &\leq \int \Psi^2 \frac{1}{2} (\rho^1 + \rho^2) d\tau \\ &= \frac{1}{2} \int \left[\Psi(v, v_*, \omega)^2 + \Psi(T(v, v_*, \omega))^2 \right] B(v, v_*, \omega) d\omega d\mu(v) d\mu(v_*) \end{aligned}$$

\square

3.2. The collision rate equation. In this section we will consider an analogue of the continuity equation for a curve of measures on \mathbb{R}^d . Instead of being driven by a vector field, the evolution will be governed by measures \mathcal{U} that prescribe the rate at which collisions happen between the particles.

More precisely, we introduce the *collision rate equation*

$$\partial_t \mu_t + \bar{\nabla} \cdot \mathcal{U}_t = 0 \quad \text{on } (0, T) \times \mathbb{R}^d. \quad (3.7)$$

Here $(\mu_t)_{t \in [0, T]}$ and $(\mathcal{U}_t)_{t \in [0, T]}$ are Borel families of measures in $\mathcal{M}_+(\mathbb{R}^d)$ and $\mathcal{M}(G)$ respectively such that

$$\int_0^T \mu_t(\mathbb{R}^d) dt < \infty, \quad (3.8)$$

$$\int_0^T |\mathcal{U}_t|(G) dt < \infty. \quad (3.9)$$

We suppose that (3.7) holds in the sense of distributions. More precisely, we require that for all $\varphi \in C_c^\infty((0, T) \times \mathbb{R}^d)$:

$$\int_0^T \int \partial_t \varphi(v) \mu_t(dv) dt + \frac{1}{4} \int_0^T \int \bar{\nabla} \varphi(v, v_*, \omega) d\mathcal{U}_t(v, v_*, \omega) dt = 0, \quad (3.10)$$

with the notation $\bar{\nabla} \varphi = \varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*)$. Note that the integrability condition (3.9) ensures that the second term is well-defined. The measures \mathcal{U}_t will be called *collision rates*.

The following is an adaptation of [1, Lemma 8.1.2].

Lemma 3.6. *Let $(\mu_t)_{t \in [0, T]}$ and $(\mathcal{U}_t)_{t \in [0, T]}$ be Borel families of measures in $\mathcal{M}_+(\mathbb{R}^d)$ and $\mathcal{M}(G)$ satisfying (3.7) and (3.8), (3.9). Then there exists a unique weakly* continuous curve $(\tilde{\mu}_t)_{t \in [0, T]}$ such that $\tilde{\mu}_t = \mu_t$ for a.e. $t \in [0, T]$. Moreover, for every $\varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and all $0 \leq t_0 \leq t_1 \leq T$ we have :*

$$\int \varphi_{t_1} d\tilde{\mu}_{t_1} - \int \varphi_{t_0} d\tilde{\mu}_{t_0} = \int_{t_0}^{t_1} \int \partial_t \varphi d\mu_t dt + \frac{1}{4} \int_{t_0}^{t_1} \int \bar{\nabla} \varphi d\mathcal{U}_t dt. \quad (3.11)$$

Proof. Let us set

$$V(t) := |\mathcal{U}_t|(G).$$

By assumption the function $t \mapsto V(t)$ belongs to $L^1(0, T)$. Fix $\xi \in C_c^\infty(\mathbb{R}^d)$. Using test functions of the form $\varphi(t, x) = \eta(t)\xi(x)$ with $\eta \in C_c^\infty(0, T)$ in (3.10), one can show that the map $t \mapsto \mu_t(\xi) := \int \xi d\mu_t$ belongs to $W^{1,1}(0, T)$. More precisely, the distributional derivative of $\mu_t(\xi)$ is given by

$$\dot{\mu}_t(\xi) = \frac{1}{4} \int \bar{\nabla} \xi d\mathcal{U}_t$$

for a.e. $t \in (0, T)$ and we can estimate

$$|\dot{\mu}_t(\xi)| \leq \frac{1}{4} \int |\bar{\nabla} \xi| d|\mathcal{U}_t| \leq \frac{1}{4} \|\xi\|_\infty V(t). \quad (3.12)$$

Based on (3.12) we can argue as in [1, Lemma 8.1.2] to obtain existence of a weakly* continuous representative $t \mapsto \tilde{\mu}_t$ and the validity of (3.11). \square

Now, we show that the collision rate equation conserves mass and under additional assumptions on the collision rates also momentum and energy. Let us define the momentum $m(\mu)$ and the energy $E(\mu)$ of a measure $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ via

$$m(\mu) := \int_{\mathbb{R}^d} v d\mu(v), \quad E(\mu) := \int_{\mathbb{R}^d} |v|^2 d\mu(v).$$

Lemma 3.7. *Let $(\mu_t)_{t \in [0, T]}$ and $(\mathcal{U}_t)_{t \in [0, T]}$ be Borel families of measures in $\mathcal{M}_+(\mathbb{R}^d)$ and $\mathcal{M}(G)$ satisfying (3.7) and (3.8), (3.9) and assume that (μ_t) is weakly* continuous. Then we have*

$$\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d) \quad \forall t \in [0, T]. \quad (3.13)$$

Assume moreover that the following stronger integrability condition holds.

$$\int_0^T \int (|v| |v_*| + |v'| |v'_*|) d\mathcal{U}_t(v, v_*, \omega) dt < \infty. \quad (3.14)$$

Then we have

$$m(\mu_t) = m(\mu_0), \quad E(\mu_t) = E(\mu_0) \quad \forall t \in [0, T]. \quad (3.15)$$

Proof. From Lemma 3.6 we have that (3.11) holds and we infer that for any $\xi \in C_c^\infty(\mathbb{R}^d)$ and any $t \in [0, 1]$:

$$\int \xi d\mu_t = \int \xi d\mu_0 + \frac{1}{4} \int_0^t \int \bar{\nabla} \xi d\mathcal{U}_s ds. \quad (3.16)$$

For $R > 0$ we choose functions $\xi_R \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \xi \leq 1$, $\xi = 1$ on B_R and $\|\xi\|_{C^1} \leq 1$. Since $\bar{\nabla} \xi_R \rightarrow 0$ pointwise, we infer from the bound $|\bar{\nabla} \xi_R| \leq 1$ and the integrability assumption (3.9) that

$$\left| \int_0^t \int \bar{\nabla} \xi_R d\mathcal{U}_s ds \right| \rightarrow 0.$$

Since $\mu_s(\xi_R)$ converges to $\mu_s(\mathbb{R}^d)$, this together with (3.16) implies that $\mu_t(\mathbb{R}^d) = \mu_0(\mathbb{R}^d)$. Let us now show conservation of energy. Consider the compactly supported function $\varphi_R(v) = (|v| \wedge R)^2 - R^2$. After a mollification argument we obtain from (3.16) that

$$\int \varphi_R d\mu_t = \int \varphi_R d\mu_0 + \frac{1}{4} \int_0^t \int \bar{\nabla} \varphi_R d\mathcal{U}_s ds. \quad (3.17)$$

The difference of the first two terms converges to $E(\mu_t) - E(\mu_0)$ as $R \rightarrow \infty$. Thus it suffice to show that the last term vanishes in the limit. Indeed, from the estimate $|\bar{\nabla} \varphi_R| \leq C(|v| |v_*| + |v'| |v'_*|)$ from Lemma 2.4 and the integrability assumption (3.14) we obtain

$$\left| \int_0^t \int \bar{\nabla} \varphi_R d\mathcal{U}_s ds \right| \rightarrow 0.$$

Conservation of momentum follows by a similar argument. \square

Remark 3.8. In the situation of Lemma 3.6, if we require $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ for all t , then following the argument in [1, Lemma 8.1.2] shows that the unique continuous representative $(\tilde{\mu}_t)_t$ is even continuous w.r.t. weak convergence.

In view of the previous results it makes sense to define solutions to the continuity equation in the following way.

Definition 3.9 (Collision rate equation). *We denote by \mathcal{CRE}_T the set of all pairs (μ, \mathcal{U}) satisfying the following conditions:*

$$\left\{ \begin{array}{ll} (i) & \mu : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^d) \text{ is weakly continuous ;} \\ (ii) & (\mathcal{U}_t)_{t \in [0, T]} \text{ is a Borel family of measures in } \mathcal{M}(G) ; \\ (iii) & \int_0^T |\mathcal{U}_t|(G) dt < \infty ; \\ (iv) & \text{We have in the sense of distributions:} \\ & \partial_t \mu_t + \bar{\nabla} \cdot \mathcal{U}_t = 0 . \end{array} \right. \quad (3.18)$$

Moreover, we will denote by $\mathcal{CRE}_T(\bar{\mu}_0, \bar{\mu}_1)$ the set of pairs $(\mu, \nu) \in \mathcal{CRE}_T$ satisfying in addition: $\mu_0 = \bar{\mu}_0$, $\mu_1 = \bar{\mu}_1$.

Remark 3.10. The continuity equation can also be tested against more general test functions. For instance, let $(\mu, \mathcal{U}) \in \mathcal{CRE}_1$ and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be bounded and continuous. Approximating φ with functions in $C_c^\infty(\mathbb{R}^d)$ that are uniformly bounded and using the integrability assumption (iii) in (3.18) to pass to the limit in (3.11) we obtain

$$\int \varphi d\mu_1 - \int \varphi d\mu_0 = \frac{1}{4} \int_0^1 \int \bar{\nabla} \varphi d\mathcal{U}_t dt .$$

A similar reasoning will often be used later on.

In view of Lemma 3.7 it makes sense to consider a stronger notion of collision rate equation imposing the conservation of momentum and energy.

Definition 3.11 (Collision rate equation with moments). *For $m \in \mathbb{R}^d$ and $E > 0$ we denote by $\mathcal{CRE}_T^{m, E}$ the set of pairs (μ, \mathcal{U}) satisfying (i), (ii) and (iv) of Definition 3.9 as well as*

$$\left\{ \begin{array}{ll} (iii') & \int_0^T \int (|v||v_*| + |v'||v'_*|) d|\mathcal{U}_t|(v, v_*, \omega) dt < \infty ; \\ (v) & \int v d\mu_t(v) = m, \quad \int |v|^2 d\mu_t(v) = E \quad \forall t \in [0, T] . \end{array} \right. \quad (3.19)$$

We write \mathcal{CRE}_T^* as a short hand for the class $\mathcal{CRE}_T^{0, 1}$.

Next, we note that being a solution to the collision rate equation is invariant under the action of the Ornstein–Uhlenbeck semigroup. To this end we first define the action of the semigroup on measures. Given $\mu \in \mathcal{P}(\mathbb{R}^d)$, we define its convolution with the Maxwellian M as usual as the measure $\mu * M \in \mathcal{P}(\mathbb{R}^d)$ given by

$$(\mu * M)(dv) = \int_{\mathbb{R}^d} M(v - w) \mu(dw) dv . \quad (3.20)$$

The action of the Ornstein–Uhlenbeck semigroup is then given by $S_t \mu = \mu_{e^{-2t}} * M_{1-e^{-2t}}$, where μ_λ is the image of μ under the map $v \mapsto \lambda v$. Of course, if $\mu = f\mathcal{L}$, we have that $\mu_\lambda = f_\lambda \mathcal{L}$ as well as $\mu * M = (f * M)\mathcal{L}$ and $S_t \mu = (S_t f)\mathcal{L}$.

Given $\mathcal{U} \in \mathcal{M}(\mathbb{R}^{2d} \times S^{d-1})$ we define its convolution $\mathcal{U} * M$ with the Maxwellian M in \mathbb{R}^{2d} as the measure given by

$$(\mathcal{U} * M)(dX, d\omega) = \int_{\mathbb{R}^{2d}} M(X - Y) \mathcal{U}(dY, d\omega) dX . \quad (3.21)$$

The action of the semigroup S_t is defined via $S_t \mathcal{U} = \mathcal{U}_{e^{-2t}} * M_{1-e^{-2t}}$, where \mathcal{U}_λ is the image of \mathcal{U} under the map $(X, \omega) \mapsto (\lambda X, \omega)$. Again, if $\mathcal{U}(dX, d\omega) = U(X, \omega) dX d\omega$ for a suitable function U , then \mathcal{U}_λ , $\mathcal{U} * M$ and $S_t \mathcal{U}$ have

densities U_λ , $U * M$ and $S_t U$ respectively, where the operations on U are defined as above, considering U for fixed ω as a function on \mathbb{R}^{2d} .

Lemma 3.12. *Let $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ and set $\mu_s^t := S_t \mu_s$, $\mathcal{U}_s^t := S_t \mathcal{U}_s$ for $t \geq 0$ and $s \in [0, T]$. Then we have $(\mu^t, \mathcal{U}^t) \in \mathcal{CRE}_T$. The same holds, when \mathcal{CRE}_T is replaced with $\mathcal{CRE}_T^{m,E}$.*

Proof. It suffices to check that being a solution to the collision rate equation is stable under scaling and convolution with M . Using the weak formulation (3.10) one readily checks that $(\mu_\lambda, \mathcal{U}_\lambda) \in \mathcal{CRE}_T$ for all $\lambda \geq 0$. To check stability under convolution fix a test function φ and set $\Phi(X) := \varphi(v) + \varphi(v_*)$. Then, using (2.7), we find

$$\begin{aligned} \frac{d}{dt} \int \varphi d(\mu_t * M) &= \frac{d}{dt} \int (\varphi * M) d\mu_t = \int \bar{\nabla}(\varphi * M) d\mathcal{U}_t \\ &= \int (\Phi * M)(T_\omega X) - (\Phi * M)(X) d\mathcal{U}_t(X, \omega) \\ &= \int ((T_\omega \Phi) * M)(X) - (\Phi * M)(X) d\mathcal{U}_t(X, \omega) \\ &= \int \Phi(T_\omega X) - \Phi(X) d(\mathcal{U}_t * M)(X, \omega) = \int \bar{\nabla} \varphi d(\mathcal{U}_t * M), \end{aligned}$$

which shows that $(\mu * M, \mathcal{U} * M) \in \mathcal{CRE}_T$. The corresponding statement for $\mathcal{CRE}_T^{m,E}$ follows in the same way, noting that S_t preserves momentum and energy. \square

The following result will allow us to extract subsequential limits from sequences of solutions to the continuity equation which have bounded action.

Proposition 3.13 (Compactness of solutions with bounded action). *In addition to Assumption 2.1, assume that*

$$\sup_k \int_A B(k, \omega) d\omega \rightarrow 0 \quad \text{when } |A| \rightarrow 0. \quad (3.22)$$

Let (μ^n, \mathcal{U}^n) be a sequence in \mathcal{CRE}_T such that $(\mu_0^n)_n$ is tight and

$$\sup_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n) dt < \infty. \quad (3.23)$$

Then there exists a couple $(\mu, \mathcal{U}) \in \mathcal{CRE}_T$ such that up to extraction of a subsequence

$$\begin{aligned} \mu_t^n &\rightharpoonup \mu_t \quad \text{weakly in } \mathcal{P}(\mathbb{R}^d) \text{ for all } t \in [0, T], \\ \mathcal{U}^n &\rightharpoonup^* \mathcal{U} \quad \text{weakly}^* \text{ in } \mathcal{M}(G \times [0, T]). \end{aligned}$$

Moreover, along this subsequence we have :

$$\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t) dt \leq \liminf_n \int_0^T \mathcal{A}(\mu_t^n, \mathcal{U}_t^n) dt.$$

Finally, if we assume in addition that $(\mu^n, \mathcal{U}^n) \in \mathcal{CRE}_T^{m,E}$ and that $v \mapsto |v|^2$ is uniformly integrable w.r.t. μ_0^n , then the conclusion holds also without the assumption (3.22) and we have that the subsequential limit pair (μ, \mathcal{U}) belongs to $\mathcal{CRE}_T^{m,E}$.

Proof. Define the measure $\mathcal{U}^n \in \mathcal{M}(G \times (0, T))$ given by $d\mathcal{U}^n(X, \omega, t) = \mathcal{U}_t^n(dX, d\omega)dt$. By Lemma 3.5 and (3.23), we obtain that for every measurable function Ψ on $\mathbb{R}^{2d} \times S^{d-1} \times [0, T]$ that is symmetric under the change of variables T we have

$$\begin{aligned} & \sup_n \int \Psi d|\mathcal{U}^n| \\ & \leq \sqrt{A} \sup_n \left(\int \Psi(X, \omega, t)^2 B(v - v_*, \omega) d\omega d\mu_t^n(v) d\mu_t^n(v_*) dt \right)^{\frac{1}{2}}, \end{aligned} \quad (3.24)$$

where A denotes the supremum in (3.23). Choosing Ψ as indicator functions of sets $G \times I$ and using Assumption 2.1, we obtain that \mathcal{U}^n has uniformly bounded total variation and more precisely we have $|\mathcal{U}^n|(G \times I) \leq C_B A \cdot \text{Leb}(B)$. Hence, we can extract a subsequence (still indexed by n) such that $\mathcal{U}^n \rightharpoonup^* \mathcal{U}$ in $\mathcal{M}(G \times [0, T])$. Moreover, the same argument shows that \mathcal{U} can be disintegrated w.r.t. Lebesgue measure on $[0, T]$ and we can write $\mathcal{U} = \int_0^T \mathcal{U}_t dt$ for a Borel family (\mathcal{U}_t) . Applying (3.24) with $\Psi \equiv 1$ shows that \mathcal{U}_t still satisfies (3.9).

Let $0 \leq t_0 \leq t_1 \leq T$ and $\xi \in C_c^\infty(\mathbb{R}^d)$. We claim that

$$\int_{t_0}^{t_1} \int \bar{\nabla} \xi d\mathcal{U}_t^n dt \xrightarrow{n \rightarrow \infty} \int_{t_0}^{t_1} \int \bar{\nabla} \xi d\mathcal{U}_t dt. \quad (3.25)$$

Note, that $\mathbf{1}_{(t_0, t_1)} \bar{\nabla} \xi$ is not continuous and not compactly supported in $G \times [0, T]$. Thus, we argue by approximation. The convergence (3.25) holds if we replace $\mathbf{1}_{(t_0, t_1)} \bar{\nabla} \xi$ by the continuous and compactly supported function $\zeta_R(t) \varphi_R(v, v_*) \cdot \mathbf{1}_{(t_0, t_1)} \bar{\nabla} \xi(v, v_*, \omega)$ where ζ_R is any continuous compactly supported function such that $0 \leq \zeta_R \leq 1$ and $\zeta_R = 1$ on $[t_0 + R^{-1}, t_1 - R^{-1}]$ and φ_R is a continuous compactly supported function such that $0 \leq \varphi_R \leq 1$ and $\varphi_R = 1$ on $B_R \subset \mathbb{R}^{2d}$. We find

$$\left| \int_{t_0}^{t_1} \int (1 - \zeta_R \varphi_R) \bar{\nabla} \xi d\mathcal{U}_t^n dt \right| \leq \int_{S_R^1 \cup S_R^2} |\bar{\nabla} \xi| d|\mathcal{U}^n|,$$

where $S_R^1 = [t_0, t_1] \times B_R^c \times S^{d-1}$ and $S_R^2 = ([t_0, t_1] \setminus [t_0 + R^{-1}, t_1 - R^{-1}]) \times \mathbb{R}^{2d} \times S^{d-1}$. The integral over S_R^2 can be estimated arguing as before and taking into account Assumption 2.1 by

$$\begin{aligned} \|\xi\|_\infty |\mathcal{U}^n|(S_R^2) & \leq \|\xi\|_\infty \sqrt{A} \left(\int_{S_R^2} B(v - v_*, \omega) d\omega d\mu_t^n(v) d\mu_t^n(v_*) dt \right)^{\frac{1}{2}}, \\ & \leq \|\xi\|_\infty \sqrt{A} C_B \frac{2}{R}. \end{aligned}$$

which vanishes as $R \rightarrow \infty$. To estimate the integral over S_R^1 , let W_R denote the set of $\omega \in S^{d-1}$ such that there exist $(v, v_*) \in B_R^c$ with $\bar{\nabla} \xi(v, v_*, \omega) \neq 0$. Note that since ξ is compactly supported, the measure of W_R goes to zero

as $R \rightarrow \infty$. We obtain a bound

$$\begin{aligned} \int_{S_R^1} |\bar{\nabla} \xi| d|\mathcal{U}^n| &\leq \|\xi\|_\infty |\mathcal{U}^n|([t_0, t_1] \times B_R^c \times W_R) \\ &\leq \|\xi\|_\infty \sqrt{A} \left(\int_{t_0}^{t_1} \int_{\mathbb{R}^{2d}} \int_{W_R} B(v - v_*, \omega) d\omega d\mu_t^n(v) d\mu_t^n(v_*) dt \right)^{\frac{1}{2}} \\ &\leq \|\xi\|_\infty \sqrt{A} \sqrt{t_1 - t_0} \left(\sup_z \int_{W_R} B(z, \omega) d\omega \right)^{\frac{1}{2}}. \end{aligned}$$

By (3.22) the last expression goes to zero as $R \rightarrow \infty$.

After extraction of another subsequence we can assume $\mu_0^n \rightharpoonup \mu_0$ weakly for some $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Using this, the convergence (3.25) and the collision rate equation in the form (3.11) for the choice $\varphi(t, v) = \xi(v)$ and $t_0 = 0, t_1 = t$ we infer that μ_t^n converges weakly* to some finite non-negative measure $\mu_t \in \mathcal{M}_+(\mathbb{R}^d)$ for every $t \in [0, T]$. From Lemma 3.7 we infer that μ_t is a probability measure for all $t \in [0, T]$. It is now easily checked that the couple (μ, \mathcal{U}) belongs to $\mathcal{CR}\mathcal{E}_T$. As in Lemma 3.3 the lower semicontinuity follows from Proposition 3.4 by considering $\int_0^T \mathcal{A}(\mu_t, \mathcal{U}_t) dt$ as an integral functional on the space $\mathcal{M}(G \times [0, T])$.

Finally, let us assume that $(\mu^n, \mathcal{U}^n) \in \mathcal{CR}\mathcal{E}_T^{m, E}$ and $v \mapsto |v|^2$ is uniformly integrable w.r.t. μ_0^n . The latter implies that

$$E(\mu_0) = \int |v|^2 d\mu_0(v) = \lim_n \int |v|^2 d\mu_0^n(v) = E.$$

Similarly, $m(\mu_0) = \lim_n m(\mu_0^n) = m$. Existence of a limit curve is obtained by arguing as before, except that we estimate the integral over S_R^1 by

$$\begin{aligned} \int_{S_R^1} |\bar{\nabla} \xi| d|\mathcal{U}^n| &\leq \|\xi\|_\infty |\mathcal{U}^n|([t_0, t_1] \times B_R^c \times S^{d-1}) \\ &\leq \|\xi\|_\infty \sqrt{A} \left(\int_{t_0}^{t_1} \int_{B_R^c} \int_{S^{d-1}} B(v - v_*, \omega) d\omega d\mu_t^n(v) d\mu_t^n(v_*) dt \right)^{\frac{1}{2}} \\ &\leq \|\xi\|_\infty \sqrt{A} \sqrt{t_1 - t_0} C_B \frac{\sqrt{E}}{R}, \end{aligned}$$

where we have used that $\mu_t^n(\{|v| \geq R\}) \leq \int \frac{|v|^2}{R^2} d\mu_t^n(v) = E/R^2$.

We use once more (3.24), this time with $\Psi = (|v||v_*| + |v'||v'_*|)$. Since μ_t^n has constant second moment E and thus uniformly bounded first moment we deduce, taking into account Assumption 2.1, that $\sup_n \int_0^T \int (|v||v_*| + |v'||v'_*|) d|\mathcal{U}_t|(v, v_*, \omega) dt < \infty$. By lower semicontinuity of moments we have that the limit \mathcal{U} satisfies (iii') of Definition 3.11. Using Lemma 3.7, we conclude that $m(\mu_t) = m(\mu_0) = m$ and $E(\mu_t) = E(\mu_0) = E$ for all $t \in [0, T]$. Thus, $(\mu, \mathcal{U}) \in \mathcal{CR}\mathcal{E}_T^{m, E}$. \square

3.3. Construction and properties of the collision distance. In this section we define the distance \mathcal{W}_B on $\mathcal{P}_*(\mathbb{R}^d)$. We will establish various properties, in particular existence of geodesics. Moreover, we will characterize absolutely continuous curves in the metric space $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$.

Definition 3.14. For $\mu_0, \mu_1 \in \mathcal{P}_*(\mathbb{R}^d)$ we define

$$\mathcal{W}_B(\mu_0, \mu_1)^2 := \inf \left\{ \int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t) dt : (\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1) \right\}. \quad (3.26)$$

Remark 3.15. In the same way one could construct a (a priori smaller) distance on the full space $\mathcal{P}(\mathbb{R}^d)$ by dropping the moments conditions and minimizing over $(\mu, \mathcal{U}) \in \mathcal{CRE}_1$ instead of \mathcal{CRE}_1^* . We will not consider this possibility here.

Let us give an equivalent characterization of the infimum in (3.26).

Lemma 3.16. For any $T > 0$ and $\mu_0, \mu_1 \in \mathcal{P}_*(\mathbb{R}^d)$ we have :

$$\mathcal{W}_B(\mu_0, \mu_1) = \inf \left\{ \int_0^T \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)} dt : (\mu, \mathcal{U}) \in \mathcal{CRE}_T^*(\mu_0, \mu_1) \right\}.$$

Proof. This follows from a standard reparametrization argument. See [1, Lem. 1.1.4] or [7, Thm. 5.4] for details in similar situations. \square

The next result shows that the infimum in the definition above is in fact a minimum.

Proposition 3.17. Let $\mu_0, \mu_1 \in \mathcal{P}_*(\mathbb{R}^d)$ be such that $W := \mathcal{W}_B(\mu_0, \mu_1)$ is finite. Then the infimum in (3.26) is attained by a curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1)$ satisfying $\mathcal{A}(\mu_t, \mathcal{U}_t) = W^2$ for a.e. $t \in [0, 1]$.

Proof. Existence of a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1)$ follows immediately by the direct method taking into account Proposition 3.13. Invoking Lemma 3.16 and Jensen's inequality we see that this curve satisfies

$$\int_0^1 \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)} dt \geq W = \left(\int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t) dt \right)^{\frac{1}{2}} \geq \int_0^1 \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)} dt.$$

Hence we must have $\mathcal{A}(\mu_t, \mathcal{U}_t) = W^2$ for a.e. $t \in [0, T]$. \square

We have the following properties of the function \mathcal{W}_B .

Theorem 3.18. \mathcal{W}_B defines a (pseudo-) metric on $\mathcal{P}_*(\mathbb{R}^d)$. The topology it induces is stronger than the weak topology and bounded sets w.r.t. \mathcal{W}_B are weakly compact. Moreover, the map $(\mu_0, \mu_1) \mapsto \mathcal{W}_B(\mu_0, \mu_1)$ is lower semicontinuous w.r.t. weak convergence. For each $\tau \in \mathcal{P}_*(\mathbb{R}^d)$ the set $\mathcal{P}_\tau := \{\mu \in \mathcal{P}_*(\mathbb{R}^d) : \mathcal{W}_B(\mu, \tau) < \infty\}$ equipped with the distance \mathcal{W}_B is a complete geodesic space.

Proof. Symmetry of \mathcal{W}_B is obvious from the fact that $\alpha(w, \cdot, \cdot) = \alpha(-w, \cdot, \cdot)$. Equation (3.11) from Lemma 3.6 shows that two curves in \mathcal{CRE}_1^* can be concatenated to obtain a curve in \mathcal{CRE}_2^* . Hence the triangle inequality follows easily using Lemma 3.16. To see that $\mathcal{W}_B(\mu_0, \mu_1) > 0$ whenever $\mu_0 \neq \mu_1$ assume that $\mathcal{W}_B(\mu_0, \mu_1) = 0$ and choose a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1)$. Then we must have $\mathcal{A}(\mu_t, \mathcal{U}_t) = 0$ and hence $\mathcal{U}_t = 0$ for a.e. $t \in (0, 1)$. From the continuity equation in the form (3.11) we infer $\mu_0 = \mu_1$. The compactness assertion and lower semicontinuity of \mathcal{W}_B follow immediately from Proposition 3.13. These in turn imply that the topology induced by \mathcal{W}_B is stronger than the weak one.

Let us now fix $\tau \in \mathcal{P}_*(\mathbb{R}^d)$ and let $\mu_0, \mu_1 \in \mathcal{P}_\tau$. By the triangle inequality we have $\mathcal{W}_B(\mu_0, \mu_1) < \infty$ and hence Proposition 3.17 yields existence of a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1)$. The curve $t \mapsto \mu_t$ is then a constant speed geodesic in \mathcal{P}_τ since it satisfies

$$\mathcal{W}_B(\mu_s, \mu_t) = \int_s^t \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)} dr = (t-s) \mathcal{W}_B(\mu_0, \mu_1) \quad \forall 0 \leq s \leq t \leq 1 .$$

To show completeness let $(\mu^n)_n$ be a Cauchy sequence in \mathcal{P}_τ . In particular the sequence is bounded w.r.t. \mathcal{W}_B and we can find a subsequence (still indexed by n) and $\mu^\infty \in \mathcal{P}_*(\mathbb{R}^d)$ such that $\mu^n \rightharpoonup \mu^\infty$ weakly. Invoking lower semicontinuity of \mathcal{W}_B and the Cauchy condition we infer that $\mathcal{W}_B(\mu^n, \mu^\infty) \rightarrow 0$ as $n \rightarrow \infty$ and that $\mu^\infty \in \mathcal{P}_\tau$. \square

It is yet unclear when precisely the distance \mathcal{W}_B is finite. However, we will see in the next section that the distance is finite for instance along solution to the Boltzmann equation. The following result shows that the distance \mathcal{W}_B can be bounded from below by the L^1 -Wasserstein distance. Recall that the L^1 -Wasserstein distance is defined for $\mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d)$ by

$$W_1(\mu_0, \mu_1) := \inf_{\pi} \int |x - y| \pi(dx, dy) ,$$

where the infimum is taken over all probability measures $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ whose first and second marginal are μ_0 and μ_1 respectively.

Proposition 3.19. *For any $\mu_0, \mu_1 \in \mathcal{P}_*(\mathbb{R}^d)$ we have the bound*

$$W_1(\mu_0, \mu_1) \leq \sqrt{2C_B} \mathcal{W}_B(\mu_0, \mu_1) .$$

Proof. We can assume that $\mathcal{W}_B(\mu_0, \mu_1) < \infty$. Take a minimizing curve $(\mu, \mathcal{U}) \in \mathcal{CRE}_1^*(\mu_0, \mu_1)$ and let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded 1-Lipschitz function. This implies that $|\bar{\nabla} \varphi| \leq 2|v - v_*|$. Taking into account Remark 3.10 and using Lemma 3.5, we estimate

$$\begin{aligned} & \left| \int \varphi d\mu_1 - \int \varphi d\mu_0 \right| \\ &= \frac{1}{4} \left| \int_0^1 \int \bar{\nabla} \varphi d\mathcal{U}_t dt \right| \\ &\leq \frac{1}{2} \int_0^1 \int |v - v_*| d|\mathcal{U}_t| (v, v_*, \omega) dt \\ &\leq \left(\int_0^1 \mathcal{A}(\mu_t, \mathcal{U}_t) dt \right)^{\frac{1}{2}} \left(\int_0^1 \int |v - v_*|^2 B(v - v_*, \omega) \mu_t(dv) \mu_t(dv_*) dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{2C_B} \mathcal{W}_B(\mu_0, \mu_1) . \end{aligned}$$

Here we have also used (2.1) and the fact that μ_t has unit variance in the last inequality. Taking the supremum over all bounded 1-Lipschitz functions φ yields the claim by Kantorovich–Rubinstein duality (see [19, Thm. 5.10, 5.16]). \square

We now give a characterization of absolutely continuous curves with respect to \mathcal{W}_B .

Proposition 3.20 (Metric velocity). *A curve $(\mu_t)_{t \in [0, T]}$ is absolutely continuous with respect to \mathcal{W}_B if and only if there exists a Borel family $(\mathcal{U}_t)_{t \in [0, T]}$ such that $(\mu, \mathcal{U}) \in \mathcal{CRE}_T^*$ and*

$$\int_0^T \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)} dt < \infty .$$

In this case, the metric derivative is bounded as $|\dot{\mu}|^2(t) \leq \mathcal{A}(\mu_t, \mathcal{U}_t)$ for a.e. $t \in [0, T]$. Moreover, there exists a unique Borel family $\tilde{\mathcal{U}}_t$ with $(\mu, \tilde{\mathcal{U}}) \in \mathcal{CRE}_T^$ such that*

$$|\dot{\mu}|^2(t) = \mathcal{A}(\mu_t, \tilde{\mathcal{U}}_t) \quad \text{for a.e. } t \in [0, T] . \quad (3.27)$$

Proof. The proof follows from the very same arguments as in [7, Thm. 5.17]. \square

We can describe the optimal velocity measures $\tilde{\mathcal{U}}_t$ appearing in the preceding proposition in more detail. We define T_μ to be the set of all $\mathcal{U} \in \mathcal{M}(G)$ such that $\mathcal{A}(\mu, \mathcal{U}) < \infty$ and $\mathcal{A}(\mu, \mathcal{U}) \leq \mathcal{A}(\mu, \mathcal{U} + \boldsymbol{\eta})$ for all $\boldsymbol{\eta} \in \mathcal{M}(G)$ satisfying

$$\frac{1}{4} \int_G \bar{\nabla} \xi d\boldsymbol{\eta} = 0 \quad \forall \xi \in C_c^\infty(G) .$$

Corollary 3.21. *Let $(\mu, \mathcal{U}) \in \mathcal{CRE}_T^*$ such that the curve $t \mapsto \mu_t$ is absolutely continuous w.r.t. \mathcal{W}_B . Then \mathcal{U} satisfies (3.27) if and only if $\mathcal{U}_t \in T_{\mu_t}$ for a.e. $t \in [0, T]$.*

If μ is absolutely continuous with respect to Lebesgue measure \mathcal{L} we can give an explicit description of T_μ . Recall that $\mathcal{B} \in \mathcal{M}(G)$ is the measure given by $d\mathcal{B}(v, v_*, \omega) = B(v - v_*, \omega) dv dv_* d\omega$.

Proposition 3.22. *Let $\mu = fm \in \mathcal{P}_*(\mathbb{R}^d)$. Then we have $\mathcal{U} \in T_\mu$ if and only if $\mathcal{U} = U\Lambda(f)\mathcal{B}$ is absolutely continuous w.r.t. the measure $\Lambda(f)\mathcal{B}$ and*

$$U \in \overline{\{\bar{\nabla} \varphi \mid \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\Lambda(f)\mathcal{B})} =: T_f .$$

Proof. If $\mathcal{A}(\mu, \mathcal{U})$ is finite we infer from Lemma 3.2 that $\mathcal{U} = U\Lambda(f)\mathcal{B}$ for some density $U : G \rightarrow \mathbb{R}$ and that $\mathcal{A}(\mu, \mathcal{U}) = \|U\|_{L^2(\Lambda(f)\mathcal{B})}^2$. Now the optimality condition in the definition of T_μ is equivalent to

$$\|U\|_{L^2(\Lambda(f)\mathcal{B})} \leq \|U + V\|_{L^2(\Lambda(f)\mathcal{B})} \quad \forall V \in N_f ,$$

where $N_f := \{V \in L^2(\Lambda(f)\mathcal{B}) : \int \bar{\nabla} \xi V \Lambda(f)\mathcal{B} = 0 \ \forall \xi \in C_c^\infty(\mathbb{R}^d)\}$. This implies the assertion of the proposition after noting that N_f is the orthogonal complement in L^2 of T_f . \square

In the light of the formal Riemannian interpretation of the distance \mathcal{W}_B one should view T_μ as the tangent space to $\mathcal{P}_*(\mathbb{R}^d)$ at the measure μ . This is reminiscent of Otto's Riemannian interpretation of the L^2 -Wasserstein space [15].

4. THE HOMOGENEOUS BOLTZMANN EQUATION AS A GRADIENT FLOW

Throughout this and the following sections we make the following assumption on the collision kernel.

Assumption 4.1. *In addition to Assumption 2.1, there exist constants $c_1, c_2 > 0$ such that for all $k \in \mathbb{R}^d, \omega \in S^{d-1}$:*

$$c_1 \leq B(k, \omega) \leq c_2. \quad (4.1)$$

Under this assumption, we show in this section that the homogeneous Boltzmann equation is the gradient flow of the entropy w.r.t. the collision distance \mathcal{W}_B . We work in the framework of metric gradient flows outlined in Section 2.4. The relevant metric space will be $(\mathcal{P}_\tau, \mathcal{W}_B)$ for a suitable τ and the functional we consider is the Boltzmann–Shannon entropy \mathcal{H} .

Recall from Proposition 3.20 that the metric derivative of an absolutely continuous curve in $(\mu_t)_t$ in $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$ is given by

$$|\dot{\mu}|(t) = \sqrt{\mathcal{A}(\mu_t, \mathcal{U}_t)},$$

for a.e. t , where \mathcal{U}_t is the optimal collision rate.

We first establish a chain rule allowing to take derivatives of the entropy along suitable absolutely continuous curves.

Proposition 4.2 (Chain rule). *Let $\mu \in AC^2((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$ such that $\mathcal{H}(\mu_t)$ is finite for some $t \in [0, T]$ and*

$$\int_0^T \sqrt{D(\mu_t)} |\dot{\mu}|(t) dt < \infty. \quad (4.2)$$

Then $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0, T]$ and we have that

$$\mathcal{H}(\mu_t) - \mathcal{H}(\mu_s) = \int_s^t \frac{1}{4} \int \bar{\nabla} \log f_r d\mathcal{U}_r dr \quad \forall 0 \leq s \leq t \leq T, \quad (4.3)$$

where f_r is the density of μ_r and \mathcal{U}_r is the optimal collision rate for the curve $(\mu_t)_t$. In particular, the map $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and we have

$$\frac{d}{dt} \mathcal{H}(\mu_t) = \frac{1}{4} \int \bar{\nabla} \log f_t d\mathcal{U}_t \quad \text{for a.e. } t. \quad (4.4)$$

Proof. Note that by (4.2) and Lemma 3.2 we have $\mu_r = f_r dv, \mathcal{U}_r = U_r dX d\omega$ for a.e. r and suitable densities f_r, U_r . We perform a three-fold regularization procedure. First, introduce the regularized entropy functional \mathcal{H}_ε defined as follows. For $\varepsilon > 0$, define $e_\varepsilon : [0, \infty) \rightarrow \mathbb{R}$ by setting $e_\varepsilon(0) = 0$ and $e'_\varepsilon(r) = \log(\varepsilon + \max(r, \varepsilon^{-1}))$. Then we define

$$\mathcal{H}_\varepsilon(\mu) = \int e_\varepsilon(f) d\mathcal{L},$$

provided that $\mu = f\mathcal{L}$ is absolutely continuous and $\mathcal{H}_\varepsilon(\mu) = +\infty$ else. Let \mathcal{U}_t be collision rates such that $(\mu, \mathcal{U}) \in \mathcal{CRE}_T^*$ and $|\dot{\mu}|(t) = \mathcal{A}(\mu_t, \mathcal{U}_t)$ for a.e. t . We regularize the curve by the Ornstein–Uhlenbeck semigroup. For $\delta > 0$ we set $\mu_t^\delta = S_\delta \mu_t = f_t^\delta \mathcal{L}$, and $\mathcal{U}_t^\delta = S_\delta \mathcal{U}_t$. We have $d\mathcal{U}_t^\delta(X, \omega) = U_t^\delta(X, \omega) dX d\omega$

for a suitable function U_t^δ . Finally, we perform a convolution in time. For a standard mollifier η on \mathbb{R} and $\gamma > 0$ we define

$$\mu_t^{\delta,\gamma} = \int \eta(t') \mu_{t-\gamma t'}^\delta dt', \quad \mathcal{U}_t^{\delta,\gamma} = \int \eta(t') \mathcal{U}_{t-\gamma t'}^\delta dt'.$$

(For this the curves are assumed to be extended trivially by $\mu_0^\delta, \mathcal{U}_0^\delta$ on $[-\gamma, 0]$ and similarly on $[T, T + \gamma]$.) The functions $f_t^{\delta,\gamma}, U_t^{\delta,\gamma}$ are defined accordingly and are the corresponding densities. By Lemma 3.12 we have that $(\mu^\delta, \mathcal{U}^\delta) \in \mathcal{CRE}_T^*$ and by linearity of the collision rate equation also $(\mu^{\delta,\gamma}, \mathcal{U}^{\delta,\gamma}) \in \mathcal{CRE}_T^*$. Note that the function $v \mapsto g_r^{\delta,\gamma,\varepsilon}(v) := e'_\varepsilon(f_r^{\delta,\gamma})(v)$ is bounded and Lipschitz. Thus, we can use the collision rate equation to calculate

$$\frac{d}{dr} \mathcal{H}_\varepsilon(\mu_r^{\delta,\gamma}) = \int_{\mathbb{R}^d} e'_\varepsilon(f_r^{\delta,\gamma}) \partial_r f_r^{\delta,\gamma} = \frac{1}{4} \int_G \bar{\nabla} g_r^{\delta,\gamma,\varepsilon} U_r^{\delta,\gamma},$$

where the integral over G is w.r.t. the measure $dX d\omega$. Integrating between s and t we obtain

$$\mathcal{H}_\varepsilon(f_t^{\delta,\gamma}) - \mathcal{H}_\varepsilon(f_s^{\delta,\gamma}) = \int_s^t \frac{1}{4} \int_G \bar{\nabla} g_r^{\delta,\gamma,\varepsilon} U_r^{\delta,\gamma} dr.$$

We will now pass to the limit in this identity to obtain (4.3) letting $\gamma \rightarrow 0$, $\varepsilon \rightarrow 0$ and $\delta \rightarrow 0$ in this order. Consider first the right hand side. Since $g^{\delta,\gamma,\varepsilon}$ is uniformly bounded for fixed ε and $U^{\delta,\gamma}$ satisfies uniform Gaussian bounds by the Ornstein–Uhlenbeck regularization, we can pass to the limit as $\gamma \rightarrow 0$ and obtain

$$\int_s^t \frac{1}{4} \int_G \bar{\nabla} g_r^{\delta,\varepsilon} U_r^\delta dr.$$

Note that $\bar{\nabla} g_r^{\delta,\varepsilon}$ converges pointwise to $\bar{\nabla} \log f_r^\delta$ as $\varepsilon \rightarrow 0$ and has a uniform quadratic (in X) bound by the Gaussian bounds on f^δ . Appealing again to the Gaussian bound on U^δ we can pass to the limit as $\varepsilon \rightarrow 0$ and obtain

$$\int_s^t \frac{1}{4} \int_G \bar{\nabla} \log f_r^\delta U_r^\delta dr. \quad (4.5)$$

Note that $\bar{\nabla} \log f_r^\delta U_r^\delta$ converges pointwise to $\bar{\nabla} \log f_r U_r$ as $\delta \rightarrow 0$ at every r where the densities of μ_r, \mathcal{U}_r exist. To pass to the limit in the integral over G it suffices to exhibit a sequence of majorants converging in $L^1(G)$. We estimate (dropping the time parameter r in the notation)

$$\begin{aligned} |\bar{\nabla} \log f^\delta U^\delta|(X, \omega) &\leq \sqrt{|\bar{\nabla} \log f^\delta|^2 \Lambda(f^\delta)} \sqrt{\frac{|U^\delta|^2}{\Lambda(f^\delta)}}(X, \omega) \\ &= \sqrt{\log \frac{T_\omega(S_\delta F)}{S_\delta F}}(T_\omega(S_\delta F) - S_\delta F) \sqrt{\frac{|S_\delta U|^2}{\Lambda(S_\delta F, T_\omega(S_\delta F))}}(X), \end{aligned}$$

where we have set again $F(X) = f f_*$. S_δ acts as a rescaled convolution. Using the commutation relation $T_\omega(S_\delta F) = S_\delta(T_\omega F)$ from Lemma 2.3 and Jensen's inequality on the convex functions $(x, y) \mapsto \log(x/y)(x - y)$ as well as $(u, x, y) \mapsto |u|^2/\Lambda(x, y)$, we obtain from the previous estimate that

$$|\bar{\nabla} \log f^\delta U^\delta|(X, \omega) \leq \sqrt{S_\delta L_1} \sqrt{S_\delta L_2}(X, \omega),$$

where we set

$$L_1(X, \omega) = \log \frac{T_\omega F}{F} (T_\omega F - F)(X) , \quad L_2(X, \omega) = \frac{|U(X, \omega)|^2}{\Lambda(F, T_\omega F)(X)} .$$

By (4.1) we then obtain

$$|\bar{\nabla} \log f^\delta U^\delta|(X, \omega) \leq \sqrt{\frac{c_2}{c_1}} \sqrt{S_\delta K_1} \sqrt{S_\delta K_2}(X, \omega) ,$$

with $K_1 = L_1 \cdot B$ and $K_2 = L_2/B$ where we write $B(X, \omega) = B(v - v_*, \omega)$. Note that $\int_G K_1 = 4D(\mu_r)$ and $\int_G K_2 = 4\mathcal{A}(\mu_r, \mathcal{U}_r)$. By assumption these quantities are finite for a.e. $r \in [0, T]$. Thus, for a.e. r we have that $S_\delta K_i$ converges to K_i in $L^1(G)$. Hence, also our majorant $\sqrt{S_\delta K_1 S_\delta K_2}$ converges to $\sqrt{K_1 K_2}$ in $L^1(G)$. Finally, to pass to the limit in the time integral, we use the already established almost everywhere in time convergence of the space integral and exhibit a majorant similar as above:

$$\begin{aligned} \int_G \bar{\nabla} \log f_r^\delta U_r^\delta dr &\leq \left(\int_G |\bar{\nabla} \log f_r^\delta|^2 \Lambda(f_r^\delta) \right)^{\frac{1}{2}} \left(\int_G \frac{|U_r^\delta|^2}{\Lambda(f_r^\delta)} \right)^{\frac{1}{2}} \\ &\leq \sqrt{\frac{c_2}{c_1}} \left(\int_G S_\delta K_1 \right)^{\frac{1}{2}} \left(\int_G S_\delta K_2 \right)^{\frac{1}{2}} = \sqrt{\frac{c_2}{c_1}} \left(\int_G K_1 \right)^{\frac{1}{2}} \left(\int_G K_2 \right)^{\frac{1}{2}} \\ &= 4 \sqrt{\frac{c_2}{c_1}} \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)} . \end{aligned}$$

Recall that the last expression is in $L^1(s, t)$ by assumption.

Let us turn to show convergence of the left hand side. Appealing again to the uniform Gaussian bounds on $f^{\delta, \gamma}$ we can pass to the limit as $\gamma \rightarrow 0$ and then $\varepsilon \rightarrow 0$ and we are left with

$$\left| \mathcal{H}(f_s^\delta) - \mathcal{H}(f_t^\delta) \right| . \quad (4.6)$$

Assume first that $\mathcal{H}(\mu_s)$ is finite. Recall that entropy is decreasing along the Ornstein–Uhlenbeck semigroup and lower semicontinuous. As $\delta \rightarrow 0$ we thus have that $\mathcal{H}(f_t^\delta)$ increases to $\mathcal{H}(\mu_t)$. The expression (4.6) converges to the left hand side of (4.3) and $\mathcal{H}(\mu_t)$ is finite due to the boundedness of the right hand side in the limit. The whole argument presented so far starting first with s being the point where $\mathcal{H}(\mu_s) < \infty$ that is given by the assumption shows that $\mathcal{H}(\mu_t) < \infty$ for all $t \in [0, T]$. Thus (4.3) is established.

Finally, using the estimate

$$\frac{1}{4} \int_G \bar{\nabla} \log f_r d\mathcal{U}_r = \sqrt{D(\mu_r)} \sqrt{\mathcal{A}(\mu_r, \mathcal{U}_r)} , \quad (4.7)$$

that is obtained just as the one before for f_r^δ we see that $t \mapsto \mathcal{H}(\mu_t)$ is absolutely continuous and (4.4) follows. \square

As a corollary we obtain that the square root of the dissipation functional D is a strong upper gradient for the entropy. More precisely, we have the following result.

Corollary 4.3. \sqrt{D} is a strong upper gradient for \mathcal{H} on $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$, i.e. for any curve $\mu \in AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$ we have

$$|\mathcal{H}(\mu_s) - \mathcal{H}(\mu_t)| \leq \int_s^t \sqrt{D(\mu_r)} |\dot{\mu}|(r) dr \quad \forall 0 \leq s \leq t \leq T. \quad (4.8)$$

Proof. Without restriction we can assume that $\int_s^t \sqrt{D(\mu_r)} |\dot{\mu}|(r) dr < \infty$, as otherwise there is nothing to prove. This implies that μ_r has a density f_r (and hence by Lemma 3.2 \mathcal{U}_r has a density U_r) for a.e. r . We can also assume that one of the measures μ_s, μ_t has finite entropy, say μ_s . Then, the claim follows immediately from Proposition 4.2 together with the estimate (4.7). \square

We can now prove the variational characterization of the homogeneous Boltzmann equation as the gradient flow of the entropy. For convenience we recall the statement here.

Recall that by a weak solution the homogeneous Boltzmann equation we mean a family of probability densities $(f_t)_{t \geq 0}$ such that we have for all $\varphi \in C_c^\infty((0, \infty) \times \mathbb{R}^d)$:

$$\int \int_{\mathbb{R}^d} \partial_t \varphi f_t dt = -\frac{1}{4} \int \int_G \bar{\nabla} \varphi (f' f'_* - f f_*) B(v - v_*, \omega) dv dv_* d\omega dt. \quad (4.9)$$

Theorem 4.4. For any curve $(f_t)_{t \geq 0}$ curve of probability densities such that

$$\int v f_0(v) dv = 0, \quad \int |v|^2 f_0(v) dv = 1, \quad \mathcal{H}(f_0) < \infty. \quad (4.10)$$

we have that

$$J_T(f) := \mathcal{H}(f_T) - \mathcal{H}(f_0) + \frac{1}{2} \int_0^T D(f_t) + |\dot{f}|^2(t) dt \geq 0 \quad \forall T \geq 0.$$

Moreover, we have $J_T(f) = 0$ for all T if and only if $(f_t)_t$ is a weak solution to the homogeneous Boltzmann equation satisfying the integrability assumptions

$$\int_0^T \int |v|^2 f_t(v) dv dt < \infty, \quad \int_0^T D(f_t) dt < \infty \quad \forall T \geq 0. \quad (4.11)$$

In other words, in the terminology of Section 2.4, the curves of maximal slope of \mathcal{H} w.r.t. the strong upper gradient \sqrt{D} are precisely the weak solutions to the Boltzmann equation.

In the present setting and under the assumption (4.10) on the initial datum, it is well-known that there exists a unique classical solution $(f_t)_t$ to the homogeneous Boltzmann equation starting from f_0 . It satisfies (4.10) for all $t \geq 0$ and moreover

$$\int_0^T D(f_t) dt = \mathcal{H}(f_0) - \mathcal{H}(f_t).$$

In particular, (4.11) holds. Thus, the conclusion of the previous theorem can be reformulated as:

Corollary 4.5. Under the assumptions of the previous theorem, there exists a unique gradient flow (i.e. curve of maximal slope) of the entropy starting from f_0 and it coincides with the unique solution to the Boltzmann equation.

Proof of Theorem (4.4). Let $(f_t)_{t \geq 0}$ be a curve satisfying (4.10). Note that by definition $|\dot{f}| = +\infty$ unless $f_t \in \mathcal{P}_*(\mathbb{R}^d)$ for a.e. t . To show $J_T(f) \geq 0$ we can assume that f is 2-absolutely continuous and $\int_0^T D(f_t) dt < \infty$, since otherwise $J_T(f) = +\infty$. But then $J_T(f) \geq 0$ follows immediately from Corollary 4.3 and Young's inequality.

We now show that any weak solution (f_t) satisfying (4.11) satisfies $J_T(f) = 0$. Setting $\mu_t = f_t \mathcal{L}$ and

$$\mathcal{U}_t = \bar{\nabla} \log f_t \Lambda(f_t) \mathcal{B} = [(f')_t (f'_*)_t - f_t (f_*)_t] \mathcal{B},$$

we see by (4.9) that (μ, \mathcal{U}) satisfies the collision rate equation (3.7). From the bound on the second moment of f and the bound (2.1) we infer that \mathcal{U} satisfies the integrability condition (3.14) and Lemma 3.7 implies that momentum and energy of f_t are conserved, i.e. $f_t \in \mathcal{P}_*(\mathbb{R}^d)$ for all t . Thus (μ, \mathcal{U}) belongs to $\mathcal{CR}\mathcal{E}^*$. Moreover, we have that $\mathcal{A}(\mu_t, \mathcal{U}_t) = D(f_t)$ and thus by (4.11) μ is absolutely continuous with $|\dot{\mu}|(t) \leq \sqrt{D(f_t)}$. In fact, since \mathcal{U}_t is of gradient form, it is the optimal collision rate, i.e. $|\dot{\mu}|(t) = \sqrt{D(f_t)}$ by Corollary 3.21 and Proposition 3.22. Finally, the chain rule (4.3) yields that for all T

$$\mathcal{H}(\mu_T) - \mathcal{H}(\mu_0) = - \int_0^T D(f_r) dr = - \frac{1}{2} \int_0^T D(f_r) + |\dot{\mu}|^2(r) dr,$$

i.e. $J_T(f) = 0$.

Conversely, let us show that any curve (f) with $J_T(f) = 0$ is a weak solution satisfying (4.11). From (4.10) we obtain that $\mathcal{H}(\mu_t) < \infty$ for all t and hence $(f_t)_t$ is locally 2-absolutely continuous in $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$ and (4.11) holds. There exists a family \mathcal{U}_t solving the collision rate equation with f_t s.t. $|\dot{f}|(t) = \mathcal{A}(f_t, \mathcal{U}_t)$ for a.e. t . By Lemma 3.2 the measure \mathcal{U}_t has a density $U_t \Lambda(f_t) \mathcal{B}$. From the chain rule (4.3) and the Cauchy–Schwartz and Young inequalities we infer that

$$\begin{aligned} \mathcal{H}(f_T) - \mathcal{H}(f_0) &= - \int_0^T \frac{1}{4} \int_G \bar{\nabla} \log f_r U_r \Lambda(f_r) \mathcal{B} dr \\ &\geq - \int_0^T \left[\sqrt{\frac{1}{4} \int_G |\bar{\nabla} \log f_r|^2 \Lambda(f_r) \mathcal{B}} \sqrt{\frac{1}{4} \int_G |U_r|^2 \Lambda(f_r) \mathcal{B}} \right] dr \\ &\geq - \frac{1}{2} \int_0^T \left[\frac{1}{4} \int_G |\bar{\nabla} \log f_r|^2 \Lambda(f_r) \mathcal{B} + \frac{1}{4} \int_G |U_r|^2 \Lambda(f_r) \mathcal{B} \right] dr \\ &= - \frac{1}{2} \int_0^T D(f_r) + |\dot{f}|^2(r) dr. \end{aligned}$$

Since $J_T(f) = 0$ we see that the two inequalities have to be identities. But equality in the Cauchy–Schwartz and Young inequalities (applied in the Hilbert space $T_{\mu_r} \mathcal{P}_*(\mathbb{R}^d)$ with the inner product $\langle U, V \rangle = \frac{1}{4} \int_G UV \Lambda(f_r) \mathcal{B}$) implies that

$$U_r = \bar{\nabla} \log f_r \quad \text{for a.e. } r.$$

Thus, the collision rate equation for (μ, \mathcal{U}) turns into the weak formulation of the Boltzmann equation. \square

5. VARIATIONAL APPROXIMATION SCHEME

In this section, consider a time-discrete variational approximation scheme for homogeneous Boltzmann equation. Recall that we make Assumption 4.1 on the collision kernel B . The scheme is the so-called minimizing movement scheme for the metric gradient flow and can be interpreted as the implicit Euler scheme for the gradient flow equation. Given a time step $\tau > 0$ and an initial datum $\mu_0 \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$, we consider a sequence $(\mu_n^\tau)_n$ defined recursively via

$$\mu_0^\tau = \mu_0, \quad \mu_n^\tau \in \operatorname{argmin}_\nu \left[\mathcal{H}(\nu) + \frac{1}{2\tau} \mathcal{W}_B(\nu, \mu_{n-1}^\tau)^2 \right]. \quad (5.1)$$

Then we build a discrete gradient flow trajectory as the piece-wise constant interpolation $(\bar{\mu}_t^\tau)_{t \geq 0}$ given by

$$\bar{\mu}_0^\tau = \mu_0, \quad \bar{\mu}_t^\tau = \mu_n^\tau \text{ if } t \in ((n-1)\tau, n\tau]. \quad (5.2)$$

Then we have the following result.

Theorem 5.1. *For any $\tau > 0$ and $\mu_0 \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\mu_0) < \infty$ the variational scheme (5.1) admits a solution $(\mu_n^\tau)_n$. As $\tau \rightarrow 0$, for any family of discrete solutions there exists a sequence $\tau_k \rightarrow 0$ and a locally 2-absolutely continuous curve $(\mu_t)_{t \geq 0}$ such that*

$$\bar{\mu}_t^{\tau_k} \rightharpoonup \mu_t \quad \forall t \in [0, \infty). \quad (5.3)$$

Moreover, any such limit curve is a gradient flow of the entropy, i.e. a weak solution to the Boltzmann equation satisfying (4.11).

With the knowledge that the Boltzmann equation in our setting has a unique solution, we obtain the following

Corollary 5.2. *For any $f_0 \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(f_0) < \infty$ and any sequence of time steps $\tau \rightarrow 0$ any discrete trajectory $\bar{\mu}_t^\tau$ given by (5.1) and (5.2) converges to the unique solution to the homogeneous Boltzmann equation starting from f_0 .*

With the work we have done so far, Theorem 5.1 follows basically from standard results for metric gradient flows where (5.1) is known as the minimizing movement scheme, see [1, Sec. 2,3]. Thus, we shall only sketch the arguments below.

We need two small additional ingredients. First, we note that the entropy dissipation is a lower semicontinuous functional w.r.t. weak convergence of probability measures.

Lemma 5.3. *For any sequence (μ_n) in $\mathcal{P}(\mathbb{R}^d)$ converging weakly to μ we have that*

$$D(\mu) \leq \liminf_n D(\mu_n). \quad (5.4)$$

Proof. We will rewrite D as an integral functional with convex integrand. To this end, consider the lower semicontinuous, convex and 1-homogeneous function $G : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined by $G(x, y) = \frac{1}{4}(x - y)(\log x - \log y)$

if $x, y > 0$ and $+\infty$ otherwise. As in the construction of the action, we associate to $\mu \in \mathcal{P}(\mathbb{R}^d)$ two measures μ^1, μ^2 given by (3.4). Then we have

$$D(\mu) = \int G\left(\frac{d\mu^1}{d\lambda}, \frac{d\mu^2}{d\lambda}\right) d\lambda,$$

where λ is any measure such that $\mu^i \ll \lambda$. Recall that weak convergence of μ_n to μ implies weak* convergence of μ_n^i to μ^i . Now the claim follows from Proposition 3.4. \square

Secondly, we relate the dissipation D to the metric slope $|\partial\mathcal{H}|$ of the entropy in the metric space $(\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B)$. Recall (2.13) for the definition of the metric slope.

Lemma 5.4. *For any $\mu \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\mu) < \infty$ we have that*

$$\sqrt{D(\mu)} \leq |\partial\mathcal{H}(\mu)|.$$

Proof. Let f_0 be the density of μ and consider the solution (f_t) to the homogeneous Boltzmann equation with initial datum f_0 . Setting $\mu_t = f_t \mathcal{L}$ and observing that

$$\begin{aligned} D(f) &= \lim_{t \searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{t} = \lim_{t \searrow 0} \frac{\mathcal{H}(\mu) - \mathcal{H}(\mu_t)}{\mathcal{W}_B(\mu_t, \mu)} \frac{\mathcal{W}_B(\mu_t, \mu)}{t} \\ &\leq |\partial\mathcal{H}(\mu)| |\dot{\mu}|(0) \leq |\partial\mathcal{H}(\mu)| \sqrt{D(\mu)} \end{aligned}$$

yields the claim. \square

Proof of Theorem 5.1. We verify that the present situation is consistent with the abstract setting considered in [1, Sec. 2].

We consider the metric space $(\mathcal{P}_{\mu_0}(\mathbb{R}^d), \mathcal{W}_B)$ and endow it with the weak topology σ . By Theorem 3.18, $(\mathcal{P}_{\mu_0}(\mathbb{R}^d), \mathcal{W})$ is separable and complete, \mathcal{W}_B is lower semicontinuous w.r.t. σ and induces a stronger topology. Recall from Section 2 that the entropy \mathcal{H} is bounded below on $\mathcal{P}_*(\mathbb{R}^d)$ and lower semicontinuous w.r.t. weak convergence. Moreover, $\mathcal{P}_*(\mathbb{R}^d)$ is compact w.r.t. weak convergence. Indeed, for any $\mu \in \mathcal{P}_*(\mathbb{R}^d)$ we have the trivial bound

$$\mu(B_R^c) \leq \int \frac{|v|^2}{R^2} d\mu(v) = \frac{1}{R^2},$$

which yields tightness of $\mathcal{P}_*(\mathbb{R}^d)$ and thus compactness by Prokhorov's theorem. In particular Assumptions 2.1.a,b,c of [1] are satisfied. From here we only sketch the argument, for details we refer to [1, Sec. 2,3].

Existence of solutions to the variational scheme follows immediately by the direct method taking into account compactness of $\mathcal{P}_*(\mathbb{R}^d)$, the lower bound on \mathcal{H} and lower semicontinuity of \mathcal{H} and \mathcal{W}_B . Then one introduces the De Giorgi interpolation of the discrete scheme defined for $t = (n-1)\tau + \delta \in ((n-1)\tau, n\tau]$ by

$$\tilde{\mu}_t^\tau \in \operatorname{argmin}_\nu \left[\mathcal{H}(\nu) + \frac{1}{2\delta} \mathcal{W}_B(\nu, \mu_{n-1}^\tau)^2 \right]. \quad (5.5)$$

Set for $t \in ((n-1)\tau, n\tau]$

$$\begin{aligned} |\dot{\mu}^\tau|(t) &:= \frac{1}{\tau} \mathcal{W}_B(\mu_n^\tau, \mu_{n-1}^\tau) , \\ G_\tau(t) &:= \frac{1}{\delta} \sup\{\mathcal{W}_B(\nu, \mu_{n-1}^\tau)\} , \quad t = (n-1)\tau + \delta , \end{aligned}$$

where the supremum is over all minimizers in (5.5). One can check that $G_\tau(t) \geq |\partial\mathcal{H}(\tilde{\mu}_t^\tau)| \geq \sqrt{D(\tilde{\mu}_t^\tau)}$, where the last inequality is due to Lemma 5.3. Then one obtains a discrete version of the energy dissipation inequality reading

$$\mathcal{H}(\mu_0) - \mathcal{H}(\tilde{\mu}_t^\tau) \geq \frac{1}{2} \int_0^t G_\tau^2(r) + |(\tilde{\mu}^\tau)'|^2(r) dr \quad \forall r \geq 0 . \quad (5.6)$$

Moreover, it is not hard to deduce from the variational scheme and the lower bound (2.4) on \mathcal{H} the following a priori estimates. For all $T > 0$ there exists $C > 0$ such that for all $\tau > 0$ and $n \in \mathbb{N}$ with $n\tau \leq T$ we have:

$$\begin{aligned} \frac{1}{2} \mathcal{W}_B(\mu_n^\tau, \mu_m^\tau)^2 &\leq \frac{1}{2\tau(n-m)} \sum_{i=m-1}^n \mathcal{W}_B(\mu_i^\tau, \mu_{i-1}^\tau)^2 \leq \frac{1}{2\tau(n-m)} (\mathcal{H}(\mu_0) + C) , \\ \mathcal{W}_B(\tilde{\mu}_t^\tau, \bar{\mu}_t^\tau)^2 &\leq 4\tau(\mathcal{H}(\mu_0) + C) . \end{aligned}$$

Recall that \mathcal{W}_B is bounded below by W_1 up to a constant. In particular, the curves $(\tilde{\mu}_t^\tau)_t$ are uniformly equicontinuous on $[0, T]$ w.r.t. W_1 . Since $(\mathcal{P}_*(\mathbb{R}^d), W_1)$ is compact the Arzela–Ascoli theorem yields the existence of a subsequential limit curve $(\mu_t)_t$ as $\tau \rightarrow 0$. Moreover, one has that $|\tilde{\mu}^\tau|$ converges weakly in L_{loc}^2 to a function A and $(\mu_t)_t$ is locally 2-absolutely continuous with $|\dot{\mu}|(t) \leq A(t)$. Now, one can pass to the limit in (5.6) by lower semicontinuity obtaining

$$\frac{1}{2} \int_0^t |\dot{\mu}|^2(r) + D(\mu_r) dr + \mathcal{H}(\mu_t) \leq \mathcal{H}(\mu_0) .$$

Thus, $(\mu_t)_t$ is a curve of maximal slope for the strong upper gradient \sqrt{D} . The reverse inequality follows from the strong upper gradient property. \square

6. CONSISTENCY WITH KAC’S RANDOM WALK

In this section we use the gradient flow structure to give a new and simple proof of the convergence of Kac’s random walk to the solution of the spatially homogeneous Boltzmann equation, see Theorem 1.4. Recall that we make Assumption 4.1 on the collision kernel B . We recall from Section 1.3 that Kac’s random walk is the continuous time Markov chain on

$$\mathcal{X}_N := \left\{ (v_1, \dots, v_N) \in \mathbb{R}^{dN} \mid \sum_{i=1}^N v_i = 0 , \sum_{i=1}^N |v_i|^2 = N \right\} .$$

with generator

$$Af(v) = \frac{1}{N} \int_{S^{d-1}} \sum_{i,j=1}^N [f(R_{ij}^\omega v) - f(v)] B(v_i - v_j, \omega) d\omega , \quad (6.1)$$

where $R_{ij}^\omega v = (v_1, \dots, v'_i, \dots, v'_j, \dots, v_N)$, with $v'_i = v_i - \langle v_i - v_j, \omega \rangle \omega$ and $v'_j = v_j + \langle v_i - v_j, \omega \rangle \omega$. The Markov chain is reversible with respect to the

Hausdorff measure π_N on \mathcal{X}_N . Denoting by μ_t^N the law of the chain starting μ_0 . Then its density f_t^N w.r.t. π_N satisfies Kac's master equation

$$\partial_t f_t^N = A f_t^N . \quad (6.2)$$

We will first detail the gradient flow structure of the master equation.

6.1. Gradient flow structure. Kac's random walk possesses the structure of a gradient flow in $\mathcal{P}(\mathcal{X}_N)$ of the relative entropy $\mathcal{H}(\cdot|\pi_N)$ with respect to a suitable metric on $\mathcal{P}(\mathcal{X}_N)$ as we shall now describe. For general Markov chains on finite state spaces a gradient flow structure has been discovered in [11, 12]. Here we briefly show how to extend this result to the present case of the continuous state space \mathcal{X}_N . The construction is similar as in Section 3, see also [8]. Let us stress however that for the purpose of showing consistency with the Boltzmann equation it will only be important to know that the solution $(f_t)_t$ to (6.2) satisfies the energy identity $J_T(f) = 0$, see (6.5) below.

We introduce a jump kernel on \mathcal{X}_N by setting

$$J(\mathbf{v}, d\mathbf{u}) = \frac{1}{N} \int_{S^{d-1}} \sum_{i,j} \delta_{R_{ij}^\omega \mathbf{v}}(d\mathbf{u}) B(v_i - v_j, \omega) d\omega .$$

Given a probability measure $\mu \in \mathcal{P}(\mathcal{X}_N)$ we define $\mu^1, \mu^2 \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ via

$$\mu^1(d\mathbf{v}d\mathbf{u}) = J(\mathbf{v}, d\mathbf{u})\mu(d\mathbf{v}) , \quad \mu^2(d\mathbf{v}d\mathbf{u}) = J(\mathbf{u}, d\mathbf{v})\mu(d\mathbf{u}) . \quad (6.3)$$

For a pair (μ, ν) with $\mu \in \mathcal{P}(\mathcal{X}_N)$ and $\nu \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ we define the action

$$\mathcal{A}_N(\mu, \nu) := \int_{\mathcal{X}_N \times \mathcal{X}_N} \alpha \left(\frac{d\nu}{d\lambda}, \frac{d\mu^1}{d\lambda}, \frac{d\mu^2}{d\lambda} \right) d\lambda ,$$

where α is defined in (3.3) and λ is any measure such that $\mu^i, \nu \ll \lambda$. Note that when $\mu = f\pi_N$ and $\mathcal{A}(\mu, \nu) < \infty$ we must have $d\nu(\mathbf{v}, \mathbf{u}) = \Psi(\mathbf{v}, \mathbf{u}) \Lambda(f(\mathbf{v}), f(\mathbf{u})) J(\mathbf{v}, d\mathbf{u}) \pi_N(d\mathbf{v})$ for some function $\Psi : \mathcal{X}_N \times \mathcal{X}_N \rightarrow \mathbb{R}$ and the action takes the form

$$\begin{aligned} \mathcal{A}_N(f, \Psi) &:= \mathcal{A}_N(\mu, \nu) \\ &= \frac{1}{N} \int_{\mathcal{X}_N} \sum_{i,j} \int_{S^{d-1}} |\Psi(\mathbf{v}, R_{ij}^\omega \mathbf{v})|^2 \Lambda(f(\mathbf{v}), f(R_{ij}^\omega \mathbf{v})) B(v_i - v_j, \omega) d\omega \pi_N(d\mathbf{v}) . \end{aligned}$$

see [8, Lem. 2.3]. We define a distance on $\mathcal{P}(\mathcal{X}_N)$ by setting

$$\mathcal{W}_N(\mu_0, \mu_1) := \inf_{\mu, \nu} \int_0^1 \mathcal{A}_N(\mu_t, \nu_t) dt ,$$

where the infimum is taken over all curves $(\mu_t)_{t \in [0,1]}$ connecting μ_0 to μ_1 and all $(\nu_t)_{t \in [0,1]}$ subject to the continuity equation

$$\frac{d}{dt} \int_{\mathcal{X}_N} \varphi d\mu_t - \int_{\mathcal{X}_N^2} [\varphi(\mathbf{u}) - \varphi(\mathbf{v})] d\nu_t(\mathbf{v}, \mathbf{u}) = 0 , \quad \forall \varphi \in C_b(\mathcal{X}_N) .$$

It follows from the results in [8, Thm. 4.4, Prop. 4.3], by considering J as a jump kernel on the ambient space \mathbb{R}^{dN} , that \mathcal{W}_N defines a distance and that the infimum in the definition is attained by an optimal pair (μ, ν) . We

denote by $|\dot{\mu}|_N(t)$ the metric derivative of a curve μ in $(\mathcal{P}(\mathcal{X}_N), \mathcal{W}_N)$. [8, Prop. 4.9] shows that exists a optimal \mathcal{V} such that (μ, \mathcal{V}) solves the continuity equation and

$$|\dot{\mu}|_N^2(t) = \mathcal{A}_N(\mu_t, \mathcal{V}_t) \quad \text{for a.e. } t .$$

We use the shorthand $\mathcal{H}_N(\mu) := \mathcal{H}(\mu|\pi_N)$ for the relative entropy w.r.t. π_N . We define the *entropy dissipation* of $\mu \in \mathcal{P}(\mathcal{X}_N)$ by

$$D_N(\mu) = \frac{1}{N} \int_{\mathcal{X}_N} \int_{S^{d-1}} \sum_{i,j} \left[f(R_{ij}^\omega \mathbf{v}) - f(\mathbf{v}) \right] \times \left[\log f(R_{ij}^\omega \mathbf{v}) - \log f(\mathbf{v}) \right] B(v_i - v_j, \omega) d\omega d\pi_N(\mathbf{v}) ,$$

provided $\mu = f\pi_N$ and we set $D_N(\mu) = +\infty$ if μ is not absolutely continuous. Note that along any solution f_t to the master equation (6.2) we have

$$\frac{d}{dt} \mathcal{H}_N(f_t) = -D_N(f_t) . \quad (6.4)$$

Proposition 6.1. *For any absolutely continuous curve $(\mu_t)_{t \geq 0}$ in $(\mathcal{P}(\mathcal{X}_N), \mathcal{W})$ with $\mathcal{H}_N(\mu_0) < \infty$ and $T > 0$ we have*

$$J_T^N(\mu) := \mathcal{H}_N(\mu_T) - \mathcal{H}_N(\mu_0) + \frac{1}{2} \int_0^T |\dot{\mu}|_N^2(t) + D_N(\mu_t) dt \geq 0 , \quad (6.5)$$

and $J_T^N(\mu) = 0$ holds for all $T > 0$ if and only if $\mu_t = f_t\pi_N$ where solves (6.2).

More precisely, D_N is a strong upper gradient for \mathcal{H}_N on $(\mathcal{P}(\mathcal{X}_N), \mathcal{W}_N)$ and any μ_0 with $\mathcal{H}_N(\mu_0) < \infty$ the solution $(f_t)_t$ to the master equation (6.2) is the unique curve of maximal slope.

Proof. We will focus on showing that any solution μ to the master equation (6.2) satisfies $J_T^N(\mu) = 0$ since this will be used in the sequel. The other statements can be obtained by following a similar line of reasoning as in Section 4, namely establishing a chain rule for the entropy analogous to Proposition 4.2 via a regularization argument (in fact the situation is much simpler due to linearity of the master equation).

Let $\mu_t = f_t\pi_N$ be a solution to the master equation (6.2). Note that the couple (μ_t, \mathcal{V}_t) solves the continuity equation if we choose

$$d\mathcal{V}_t(\mathbf{v}, \mathbf{u}) = \Psi_t(\mathbf{v}, \mathbf{u}) \Lambda(f_t(\mathbf{v}), f_t(\mathbf{u})) J(\mathbf{v}, d\mathbf{u}) \pi^N(d\mathbf{v})$$

with $\Psi_t(\mathbf{v}, \mathbf{u}) = \log f_t(\mathbf{u}) - \log f_t(\mathbf{v})$. Since Ψ_t is of gradient form it is in fact optimal and we have $|\dot{\mu}|_N^2(t) = \mathcal{A}(\mu_t, \mathcal{V}_t)$ for a.e. t . Note moreover that $\mathcal{A}(\mu_t, \mathcal{V}_t) = D_N(\mu_t)$. Thus integrating (6.4) yields $J_T(\mu) = 0$ for all T . \square

6.2. Convergence to the Boltzmann equation. Now we will show that the distribution of the empirical measure of N particles evolving by Kac's random walk converges to the solution of the homogeneous Boltzmann equation as $N \rightarrow \infty$.

Consider the map assigning to a configuration in \mathcal{X}_N its empirical measure

$$L_N : \mathcal{X}_N \rightarrow \mathcal{P}_*(\mathbb{R}^d) , \quad \mathbf{v} \mapsto \frac{1}{N} \sum_{i=1}^N \delta_{v_i} ,$$

where $\mathcal{P}_*(\mathbb{R}^d)$ denotes the set of probability measure on \mathbb{R}^d with zero mean and unit variance, see (2.2). For a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ we denote by $\mathcal{H}(\mu|M)$ the relative entropy with respect to the standard Gaussian, see (2.3). Recall that $\mathcal{H}(\mu|M) \geq 0$ and that

$$\mathcal{H}(\mu|M) = \mathcal{H}(\mu) + \frac{1}{2} \int |v|^2 d\mu(v) + \frac{d}{2} \log(2\pi) ,$$

provided the right hand side is defined. Thus for $\mu \in \mathcal{P}_*(\mathbb{R}^d)$ we have $\mathcal{H}(\mu|M) = \mathcal{H}(\mu) - \mathcal{H}(M)$. Moreover, it holds

$$\mathcal{H}(M) = \inf \{ \mathcal{H}(\nu) | \nu \in \mathcal{P}_*(\mathbb{R}^d) \} =: \mathcal{H}_0 .$$

We equip $\mathcal{P}_*(\mathbb{R}^d)$ with the topology of weak convergence plus convergence of the first moment (or equivalently, convergence in the L^1 -Wasserstein distance W_1), and denote by $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ the set of Borel probability measure on $\mathcal{P}_*(\mathbb{R}^d)$. We equip $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ with the weak topology of weak convergence induced by the L^1 -Wasserstein distance on $\mathcal{P}_*(\mathbb{R}^d)$. For convenience we recall the convergence statement.

Theorem 6.2. *For each N let $(\mu_t^N)_{t \geq 0}$ be a the law of Kac's random walk starting from μ_0^N and let $c_t^N := (L_N)_\# \mu_t^N \in \mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ be the law of the empirical measures. Assume that μ_0^N is well-prepared for some $\nu_0 = f_0 \mathcal{L} \in \mathcal{P}_*(\mathbb{R}^d)$ with $\mathcal{H}(\nu_0|M) < \infty$ in the sense that in the limit $N \rightarrow \infty$*

$$c_0^N \rightharpoonup \delta_{\nu_0} , \quad \frac{1}{N} \mathcal{H}_N(\mu_0^N) \rightarrow \mathcal{H}(\nu_0) - \mathcal{H}_0 .$$

Then, for all $t > 0$, c_t^N converges weakly to δ_{ν_t} , where $\nu_t = f_t \mathcal{L}$ and f_t is the unique solution to the spatially homogeneous Boltzmann equation with initial datum f_0 . Moreover, we have that

$$\frac{1}{N} \mathcal{H}_N(\mu_t^N) \rightarrow \mathcal{H}(\nu_t) - \mathcal{H}_0 . \quad (6.6)$$

We first give the proof of this theorem and then collect necessary ingredients afterwards.

Proof. By Proposition 6.1 we have that

$$J_T^N(\mu^N) = \mathcal{H}_N(\mu_T) - \mathcal{H}_N(\mu_0) + \frac{1}{2} \int_0^T |\dot{\mu}_N^2(t) + D_N(\mu_t) dt = 0 . \quad (6.7)$$

Together with the convergence of $\mathcal{H}_N(\mu_0^N)/N$ this implies in particular that

$$\sup_N \frac{1}{N} \int_0^T |\dot{\mu}_N^2(t) dt < \infty .$$

The compactness result Lemma 6.3 yields that up to a subsequence we have that $c_t^N \rightharpoonup c_t$ for all t and a curve $(c_t)_t$ in $\mathcal{P}_*(\mathbb{R}^d)$, moreover by Lemma 6.5 this curve can be represented as $c_t = (e_t)_\# \Theta$ for a probability measure Θ on $\Gamma_T := AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$. Thanks to the lim inf-inequalities for the entropy, dissipation and action given by (6.11), (6.18) and (6.17), we can now divide by N in (6.7) and pass to the limes inferior obtaining

$$\int_{\Gamma_T} \left[\mathcal{H}(\eta_T) - \mathcal{H}(\eta_0) + \frac{1}{2} \int_0^T |\dot{\eta}|^2(t) + D(\eta_t) dt \right] d\Theta(\eta) \leq 0 . \quad (6.8)$$

Since \sqrt{D} is a strong upper gradient for \mathcal{H} , the integrand is non-negative, see Corollary 4.3. Thus we have in fact equality in (6.8) and we infer that Θ is concentrated on curves of maximal slope of \mathcal{H} . Since Θ -a.s. $\eta_0 = \nu_0$ and by Corollary 4.5 the curve of maximal slope starting from ν_0 is unique and given by $\nu_t = f_t \mathcal{L}$ with f_t the unique solution to the homogeneous Boltzmann equation with initial datum f_0 , we infer that $c_t = (e_t)_\# \Theta = \delta_{\nu_t}$ for all t . By uniqueness of the limit the convergence of c_t^N to δ_{ν_t} holds for the full sequence. Finally, we prove (6.6). From the previous discussion we retain that

$$\begin{aligned} 0 = \liminf_N \frac{1}{N} J_t^N(\mu^N) &\geq \liminf_N \frac{1}{N} \mathcal{H}_N(\mu_t^N) - \lim_N \frac{1}{N} \mathcal{H}_N(\mu_0^N) \\ &\quad + \frac{1}{2} \liminf_N \frac{1}{N} \int_0^t D_N(\mu_r^N) + |\dot{\mu}^N|^2(r) dr \\ &\geq \mathcal{H}(\nu_t) - \mathcal{H}(\nu_0) + \frac{1}{2} \int_0^t D(\nu_r) + |\dot{\nu}|^2(r) dr \\ &\geq 0. \end{aligned}$$

Using that all these inequalities are equalities and again (6.11), (6.18), (6.17) we infer that we have equality

$$\liminf_N \frac{1}{N} \mathcal{H}_N(\mu_t^N) = \mathcal{H}(\nu_t) - \mathcal{H}_0.$$

Since by the same argument this must hold for any subsequence, we conclude the convergence (6.6) for the full sequence. \square

We now collect the ingredients to the previous proof. We start with a simple compactness result in Lemma 6.3. Then we establish the lim inf estimates for the entropy, dissipation and metric speed in Lemma 6.4 and Lemma 6.5. Although the proofs of the latter might seem long, the argument is in fact rather simple and boils down to the lower semicontinuity of integral functionals stated in Proposition 3.4. A non-trivial additional ingredient that we develop in Lemma 6.5 is a probabilistic representation result that allows to view certain curves in $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ as superposition of curves in $\mathcal{P}_*(\mathbb{R}^d)$.

Lemma 6.3. *Let $(\mu_t^N)_{t \in [0, T]}$ be a sequence of absolutely continuous curves in $(\mathcal{P}(\mathcal{X}_N), \mathcal{W}_N)$ such that*

$$\sup_N \frac{1}{N} \int_0^T |\dot{\mu}^N|_N^2(t) dt < \infty, \quad (6.9)$$

and put $c_t^N = (L_N)_\# \mu_t^N$. Then up to a subsequence we have that $c_t^N \rightharpoonup c_t$ weakly for all t and a curve c in $\mathcal{P}(\mathcal{P}_(\mathbb{R}^d))$.*

Proof. First note that

$$\mathcal{W}_N(\mu_s^N, \mu_t^N) \geq \frac{C}{\sqrt{N}} W_1(\mu_s^N, \mu_t^N) \geq C \sqrt{N} W_1(c_s^N, c_t^N) \quad (6.10)$$

for some constant $C > 0$. The first inequality is given by [8, Prop. 4.5] where we view μ_s^N, μ_t^N as measures on \mathbb{R}^{Nd} equipped with the distance $d(\mathbf{v}, \mathbf{u}) = \sum_i |v_i - u_i|$ and note that $\int_{\mathbb{R}^{Nd}} d(\mathbf{v}, \mathbf{u})^2 J(\mathbf{v}, d\mathbf{u}) = CN$. The W_1 -distance in the right hand side of the second inequality is induced by

the W_1 -distance on $\mathcal{P}_*(\mathbb{R}^d)$. Together with (6.10), (6.9) implies that the curves $(c_t^N)_t$ are uniformly equicontinuous in $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ w.r.t. the distance W_1 and hence also w.r.t. any smaller distance metrizing the weak topology on $\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ induced by the W_1 -distance on $\mathcal{P}_*(\mathbb{R}^d)$. Note that $(\mathcal{P}_*(\mathbb{R}^d), W_1)$ is compact due to the uniform bound on the second moment and hence $(\mathcal{P}(\mathcal{P}_*(\mathbb{R}^d)), W_1)$ is compact. Thus by the Arzela–Ascoli theorem, the curves (c_t^N) converges pointwise weakly to a curve (c_t) . \square

Lemma 6.4 (lim inf-inequality for the entropy). *Let $(\mu^N)_N$ be a sequence of measures in $\mathcal{P}(\mathcal{X}_N)$ such that $c^N = (L_N)_\# \mu^N$ converges weakly to $c \in \mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$. Then we have that*

$$\liminf_N \mathcal{H}_N(\mu^N) \geq \int_{\mathcal{P}_*(\mathbb{R}^d)} \mathcal{H}(\nu|M) \, dc(\nu) . \quad (6.11)$$

Proof. For the purpose of this proof we equip $\mathcal{P}_*(\mathbb{R}^d)$ with the topology of weak convergence plus convergence of the second moment (or equivalently, convergence in L^2 -Wasserstein distance) More precisely require convergence in duality with $C_2(\mathbb{R}^d)$ the set of continuous functions with at most quadratic growth.

(i) We put $q^N := (L_N)_\# \pi_N \in \mathcal{P}(\mathcal{P}_*(\mathbb{R}^d))$ and note that the entropy can be decomposed as follows

$$\begin{aligned} \mathcal{H}_N(\mu^N) &= \mathcal{H}(c^N|q^N) + \int_{\mathcal{P}_*(\mathbb{R}^d)} \mathcal{H}(\mu^N(\cdot|L_N = \nu)|\pi^N(\cdot|L_N = \nu)) \, dc^N(\nu) \\ &\geq \mathcal{H}(c^N|q^N) , \end{aligned} \quad (6.12)$$

where we have used that the entropy w.r.t. a probability measure is non-negative. We recall the following duality formula for the entropy

$$\begin{aligned} \mathcal{H}(c^N|q^N) &= \sup \left\{ \int_{\mathcal{P}_*(\mathbb{R}^d)} F(\nu) \, dc^N(\nu) \mid F \in C_b(\mathcal{P}_*(\mathbb{R}^d)) \text{ s.t.} \right. \\ &\quad \left. \int e^{F(\nu)} dq^N(\nu) \leq 1 \right\} . \end{aligned} \quad (6.13)$$

For $t > 0$ we define the function

$$F_t(\nu) = \inf_{\sigma \in \mathcal{P}_*(\mathbb{R}^d)} \mathcal{H}(\sigma|M) + \frac{1}{2t} W_2^2(\sigma, \nu) , \quad (6.14)$$

where W_2 denotes the L^2 -Wasserstein distance. Then F_t is continuous on $\mathcal{P}_*(\mathbb{R}^d)$ and bounded, more precisely we have:

$$0 \leq F_t(\nu) \leq \frac{1}{2t} W_2^2(M, \nu) \leq \frac{2}{t} ,$$

using that the second moment of M and ν is 1. Applying (6.12) and (6.13) with $F = NF_t - \log Z_t^N$, where

$$Z_t^N = \int e^{NF_t(\nu)} dq^N(\nu) ,$$

we obtain the bound

$$\frac{1}{N} \mathcal{H}_N(\mu^N) \geq \int F_t(\nu) dc^N(\nu) - \frac{1}{N} \log Z_t^N . \quad (6.15)$$

- (ii) Since by assumption c^N converges weakly to c as $N \rightarrow \infty$ the first term in (6.15) converges to $\int F_t(\nu) d\mathbf{c}(\nu)$. Using the fact that $F_t(\nu) \nearrow \mathcal{H}(\nu|M)$ as $t \searrow 0$, this will yield the desired bound (6.11) by monotone convergence.
- (iii) It thus remains to show the following claim:

$$\lim_N \frac{1}{N} \log Z_t^N = 0. \quad (6.16)$$

This will be accomplished by showing that q^N satisfies a large deviation principle and applying Varadhan's lemma on the evaluation of exponential integrals.

First note that the measure π^N can be realized as the law of

$$\mathbf{v} = a(\mathbf{w})^{-1}(w_1 - b(\mathbf{w}), \dots, w_N - b(\mathbf{w})) ,$$

where $b(\mathbf{w}) = \sum_{j=1}^N w_j$, $a(\mathbf{w}) = \sum_{j=1}^N (w_j - b(\mathbf{w}))^2$ and where w_1, \dots, w_N are iid standard Gaussian random vectors in \mathbb{R}^d . Let us denote by m^N the law of the empirical measures $\frac{1}{N} \sum_j \delta_{w_j}$ on $\mathcal{P}_2(\mathbb{R}^d)$. The measure q^N can thus be realized as $\Psi_{\#} m^N$ where $\Psi : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathcal{P}_*(\mathbb{R}^d)$ is the map given by $\Psi(\nu) = (a(\nu)^{-1}(\cdot - b(\nu)))_{\#} \nu$, where $b(\nu) = \int v d\nu(v)$ and $a(\nu) = \int (v - b(\nu))^2 d\nu(v)$. It can be checked that Ψ is continuous w.r.t. the topology induced by W_2 .

A version of Sanov's theorem states that the measures m^N satisfy a large deviation principle on $\mathcal{P}_*(\mathbb{R}^d)$ equipped with the W_2 -topology with good rate function $\mathcal{H}(\cdot|M)$. (To obtain this, one can follow for instance the derivation in [6, Sec. 6.2] where the corresponding result is established for the weak topology and note that the measures m^N are exponentially tight also in the topology we consider. For instance, the sets $K_\alpha = \{\nu : \int |v|^4 d\nu(v) \leq \alpha\}$ are compact and their complements have exponentially small probability under m^N).

Now, by the contraction principle [6, Thm. 4.2.1] the measures q^N satisfy a large deviation principle with good rate function

$$I(\nu) = \inf \left\{ \mathcal{H}(\nu'|M) \mid \nu' \in \mathcal{P}_2(\mathbb{R}^d), \Psi(\nu') = \nu \right\} = \mathcal{H}(\nu|M) .$$

Finally, by Varadhan's lemma [6, Thm. 4.3.1], the boundedness of F_t and the trivial bound $\mathcal{F}_t \leq \mathcal{H}(\cdot|M)$ we obtain

$$\lim_N \frac{1}{N} \log Z_t^N = \sup \left\{ F_t(\nu) - \mathcal{H}(\nu|M) \mid \nu \in \mathcal{P}_*(\mathbb{R}^d) \right\} = 0 ,$$

which finishes the proof. \square

Lemma 6.5 (Representation and lim inf-inequality for action and dissipation). *Under the assumptions of Lemma 6.3 there exists a probability measure Θ on $\Gamma_T := AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$ such that $c_t = (e_t)_{\#} \Theta$ and we have*

$$\liminf_N \frac{1}{N} \int_0^T |\dot{\mu}^N|_N^2(t) dt \geq \int_{\Gamma_T} \left[\int_0^T |\dot{\eta}|^2(t) dt \right] d\Theta(\eta) , \quad (6.17)$$

$$\liminf_N \frac{1}{N} \int_0^T D_N(\mu_t^N) dt \geq \int_{\Gamma_T} \left[\int_0^T D(\eta_t) dt \right] d\Theta(\eta) , \quad (6.18)$$

where $|\eta|$ is the metric speed w.r.t. the collision distance \mathcal{W}_B and $D(\eta)$ is the dissipation defined in (2.5).

Proof. The following notation will be very convenient. For three measures $\gamma^1, \gamma^2, \gamma^3$ on some space Y we define

$$\mathcal{F}(\gamma^1, \gamma^2, \gamma^3) = \int_Y \alpha \left(\frac{d\gamma^1}{d\sigma}, \frac{d\gamma^2}{d\sigma}, \frac{d\gamma^3}{d\sigma} \right) d\sigma ,$$

where α is defined in Sec. 3.1 and σ is any measure on Y such that $\gamma^1, \gamma^2, \gamma^3 \ll \sigma$. Note that \mathcal{F} is convex and lower semicontinuous by Proposition 3.4. Recall that we can choose measures $\mathcal{V}_t^N \in \mathcal{M}(\mathcal{X}_N \times \mathcal{X}_N)$ such that $|\dot{\mu}^N|_N^2(t) = \mathcal{A}_N(\mu_t^N, \mathcal{V}_t^N) = \mathcal{F}(\mathcal{V}_t^N, \mu_t^{N,1}, \mu_t^{N,2})$ for a.e. t . We define the measures $\tilde{\gamma}_t^N, \tilde{\beta}^{N,k}$ on $\mathcal{P}_*(\mathbb{R}^d)^2$ as follows

$$\tilde{\gamma}_t^N = \frac{1}{N}(L_N \times L_N)_\# \mathcal{V}_t^N , \quad \tilde{\beta}_t^{N,k} = \frac{1}{N}(L_N \times L_N)_\# \mu_t^{N,k}$$

and note that

$$\begin{aligned} d\tilde{\beta}_t^{N,1}(\eta, \eta') &= \int_{(\mathbb{R}^d)^2} \int_{S^{d-1}} \delta_{\eta^N, v, v_*, \omega}(d\eta') B(v - v_*, \omega) \eta(dv) \eta(dv_*) d\omega dc_t^N(\eta) , \\ d\tilde{\beta}_t^{N,2}(\eta, \eta') &= \int_{(\mathbb{R}^d)^2} \int_{S^{d-1}} \delta_{\eta^N, v', v'_*, \omega}(d\eta') B(v - v_*, \omega) \eta(dv') \eta(dv'_*) d\omega dc_t^N(\eta) , \end{aligned}$$

where we set $\eta^{N, v, v_*, \omega} = \eta - \frac{1}{N}(\delta_v + \delta_{v_*} - \delta_{v'} - \delta_{v'_*})$ with v, v_*, v', v'_* related via (1.3) and recall that $c_t^N(d\eta) = (L_N)_\# \mu_t^N(d\eta)$. By convexity of \mathcal{F} we find that

$$\frac{1}{N} |\dot{\mu}^N|_N^2(t) = \frac{1}{N} \mathcal{F}(\mathcal{V}_t^N, \mu_t^{N,1}, \mu_t^{N,2}) \geq \mathcal{F}(\tilde{\gamma}_t^N, \tilde{\beta}_t^{N,1}, \tilde{\beta}_t^{N,2}) .$$

By finiteness of the right hand side we conclude that $\tilde{\gamma}_t^N \ll \Lambda(\tilde{\beta}_t^N)$, where $d\Lambda(\tilde{\beta}_t^N) = \Lambda\left(\frac{d\tilde{\beta}_t^{N,1}}{d\sigma}, \frac{d\tilde{\beta}_t^{N,2}}{d\sigma}\right) d\sigma$. Hence we have that

$$d\tilde{\gamma}_t^N(\eta, \eta') = \int_{(\mathbb{R}^d)^2} \int_{S^{d-1}} \delta_{\eta^N, v, v_*, \omega}(d\eta') d\mathcal{U}_{\eta, t}^N(v, v_*, \omega) dc_t^N(\eta)$$

for suitable measures $\mathcal{U}_{\eta, t}^N$ on $(\mathbb{R}^d)^2 \times S^{d-1}$. We also define the measures $\gamma^N, \beta^{N,k} \in \mathcal{M}(\mathcal{P}_*(\mathbb{R}^d) \times (\mathbb{R}^d)^2 \times S^{d-1})$ by

$$\begin{aligned} d\gamma_t^N(\eta, v, v_*, \omega) &= d\mathcal{U}_{\eta, t}^N(v, v_*, \omega) dc_t^N(\eta) , \\ d\beta^{N,k}(\eta, v, v_*, \omega) &= d\eta^k(v, v_*, \omega) dc_t^N(\eta) , \end{aligned}$$

where $\eta^1, \eta^2 \in \mathcal{M}((\mathbb{R}^d)^2 \times S^{d-1})$ are given by (3.4) and set $\gamma^N, \beta^{N,k}$ by setting and $d\gamma^N = d\gamma_t^N dt$ and $d\beta^{N,k} = d\beta_t^{N,k} dt$. Then we have that $\int_0^T \mathcal{F}(\tilde{\gamma}_t^N, \tilde{\beta}_t^{N,1}, \tilde{\beta}_t^{N,2}) dt = \mathcal{F}(\gamma^N, \beta^{N,1}, \beta^{N,2})$.

Note that as $N \rightarrow \infty$ we have $\beta^{N,k} \rightharpoonup \beta^k$ with

$$d\beta^k(\eta, v, v_*, \omega, t) = d\eta^k(v, v_*, \omega) dc_t(\eta) dt .$$

Arguing as in the proof of Proposition 3.13 via an integrability estimate similar to Lemma 3.5 one can show that γ^N converges vaguely up to a subsequence to a limit γ . By lower semicontinuity of \mathcal{F} we find

$$\liminf_N \frac{1}{N} \int_s^t |\dot{\mu}^N|_N^2(r) dr \geq \liminf_N \mathcal{F}(\gamma^N, \beta^{N,1}, \beta^{N,2}) \geq \mathcal{F}(\gamma, \beta^1, \beta^2).$$

From the finiteness of the right hand side it follows that $\gamma \ll \Lambda(\frac{d\beta^1}{d\sigma}, \frac{d\beta^2}{d\sigma})d\sigma$ and hence that

$$d\gamma(\eta, v, v_*, \omega, t) = d\mathcal{U}_{\eta,t}(v, v_*, \omega) dc_t(\eta) dt,$$

for a suitable family of measures $\mathcal{U}_{\eta,t}$ and thus

$$\begin{aligned} \liminf_N \frac{1}{N} \int_0^T |\dot{\mu}^N|_N^2(t) dt &\geq \mathcal{F}(\gamma, \beta^1, \beta^2) \\ &= \int_0^T \int_{\mathcal{P}_*(\mathbb{R}^d)} \mathcal{A}(\nu_t, \mathcal{U}_{\nu_t,t}) dc_t(\eta) dt. \end{aligned} \quad (6.19)$$

We obtain a liminf estimate for the dissipation in a similar fashion. Set $G(s, t) = (t - s)(\log t - \log s)$ and recall that G is convex, lower semicontinuous and 1-homogeneous. For two measures α^1, α^2 on some space Y we define the convex and lower semicontinuous functional

$$\mathcal{G}(\alpha^1, \alpha^2) = \int_Y G\left(\frac{d\alpha^1}{d\sigma}, \frac{d\alpha^2}{d\sigma}\right) d\sigma,$$

and note that $D_N(\mu_t^N) = \mathcal{G}(\mu^{N,1}, \mu^{N,2})$. From convexity and lower semicontinuity we then obtain

$$\begin{aligned} \liminf_N \frac{1}{N} D_N(\mu_t^N) &\geq \liminf_N \mathcal{G}(\tilde{\beta}_t^{N,1}, \tilde{\beta}_t^{N,2}) = \liminf_N \mathcal{G}(\beta_t^{N,1}, \beta_t^{N,2}) \\ &\geq \mathcal{G}(\beta_t^1, \beta_t^2) = \int_{\mathcal{P}_*(\mathbb{R}^d)} D(\eta) dc_t(\eta). \end{aligned} \quad (6.20)$$

The proof will be finished once we establish the following claim concerning a representation of the curve c .

Claim 6.6. There exists a probability Θ on $\Gamma_T = AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$ such that

- (1) $c_t = (e_t)_\# \Theta$ for all t ,
- (2) For Θ -a.e. curve $(\eta_t)_{t \in [0, T]}$, $(\eta_t, \mathcal{U}_{\eta_t,t})_t$ belongs to \mathcal{CRE}_T^* .

Indeed, (2) implies that Θ -a.e. curve $(\eta_t)_t$ satisfies $|\dot{\eta}|^2(t) \leq \mathcal{A}(\eta_t, \mathcal{U}_{\eta_t,t})$ for a.e. t . By (1), (6.19) and (6.20) can be transformed into (6.17) and (6.18). To proof the claim, we will first show that the curve $(c_t)_t$ satisfies a sort of continuity equation over the space $\mathcal{P}_*(\mathbb{R}^d)$. We will then sketch how to derive the desired probabilistic representation from classical representation results for the continuity equation over Euclidean space by a finite dimensional approximation argument.

Fix a countable collection $\{f_i\}_{i \in \mathbb{N}}$ of functions that is dense (w.r.t. uniform convergence) in the set of 1-Lipschitz functions on \mathbb{R}^d vanishing at 0. Consider cylinder functions $F : \mathcal{P}_*(\mathbb{R}^d) \rightarrow \mathbb{R}$ given by

$$F(\eta) = \varphi(\langle f_1, \eta \rangle, \dots, \langle f_m, \eta \rangle),$$

with $\varphi \in C_c^\infty(\mathbb{R}^m)$, where we set $\langle f, \eta \rangle = \int f d\eta$. Write for short $X^m(\eta) = (\langle f_1, \eta \rangle, \dots, \langle f_m, \eta \rangle) \in \mathbb{R}^m$. Fix $a \in C_c^\infty(0, T)$. Using the previous notation, we obtain from the continuity equation for (μ_t^N, ν_t^N) after passing to the empirical measure:

$$\int_0^T a'(t) \int F d c_t^N dt = -N \int_0^T a(t) \int [F(\eta') - F(\eta)] d \tilde{\gamma}_t^N(\eta, \eta') dt,$$

Using that $F(\eta') - F(\eta) = \sum_i \partial_i \varphi(X^m(\eta)) \bar{\nabla} f_i(v, v_*, \omega) + o(1)$ for $\eta' = \eta + \frac{1}{N}(\delta_{v'} + \delta_{v'_*} - \delta_v - \delta_{v_*})$ with $\bar{\nabla} f_i(v, v_*, \omega) = f_i(v') + f_i(v'_*) - f_i(v) - f_i(v_*)$ and that $\int_0^T \int \mathcal{A}(\eta, \mathcal{U}_{\eta,t}^N) d c_t^N(\eta) dt$ is uniformly bounded we infer from the convergence of c_t^N to c and of $\gamma_t^N \rightarrow \gamma$ that

$$\begin{aligned} & \int_0^T a'(t) \int F d c_t dt \\ &= - \int_0^T a(t) \int \sum_i \partial_i \varphi(X^m(\eta)) \bar{\nabla} f_i(v, v_*, \omega) d \mathcal{U}_{\eta,t}(v, v_*, \omega) d c_t(\eta) dt. \end{aligned} \quad (6.21)$$

We define $\rho_t^m \in \mathcal{P}(\mathbb{R}^m)$ by $\rho_t^m = (X^m)_\# c_t$ and define a vector field $V_t^m = (V_{1,t}^m, \dots, V_{m,t}^m)$ on \mathbb{R}^m by

$$V_{i,t}^m(x) = \int_{\mathcal{P}_*(\mathbb{R}^d)} \int_{\mathbb{R}^{2d} \times S^{d-1}} \bar{\nabla} f_i d \mathcal{U}_{\eta,t} d c_t^x(\eta),$$

where c_t^x is the disintegration of c w.r.t. its marginal ρ_t^m , i.e. $c_t = \int_{\mathbb{R}^m} c_t^x d \rho_t^m(x)$. Now, (6.21) shows that (ρ_t^m, V_t^m) solves the continuity equation in \mathbb{R}^m :

$$\int_0^T a'(t) \int \varphi d \rho_t^m dt = - \int_0^T a(t) \int \nabla \varphi \cdot V_t^m d \rho_t^m dt.$$

Moreover, using Jensen's inequality and Lemma 3.5 we see that the vector field is square integrable:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^m} |V_t^m|^2 d \rho_t^m dt &= \sum_{i=1}^m \int \left| \int \bar{\nabla} f_i d \mathcal{U}_{\eta,t} d c_t^x(\eta) \right|^2 d \rho_t^m(x) \\ &\leq \sum_{i=1}^m \int \left| \int \bar{\nabla} f_i d \mathcal{U}_{\eta,t} \right|^2 d c_t(\eta) \\ &\leq 4 \sum_{i=1}^m \|f_i\|_\infty^2 \int_0^T \int \mathcal{A}(\eta, \mathcal{U}_{\eta,t}) d c_t(\eta) dt < \infty. \end{aligned}$$

Thus, by a probabilistic representation result [1, Lemma 8.2] we can find probability measures Θ^m on $AC((0, T), \mathbb{R}^m)$ such that $\rho_t^m = (e_t)_\# \Theta^m$ for all t and Θ^m a.e. curve γ is an integral curve of V^m , i.e. $\dot{\gamma}_t = V_t^m(\gamma_t)$ for a.e. t . In this result, \mathbb{R}^m is equipped with the Euclidean distance, but we will consider Θ^m as measures on absolutely continuous curves with the

weaker distance $d(x, y) = \sup_i |x_i - y_i|$. A look at the proof reveals that the measures Θ^m are consistent, i.e. for $k \leq m$ we have $\pi_{\#}^{m,k} \Theta^m = \Theta^k$ for the natural projection $\pi^{m,k} : AC((0, T), \mathbb{R}^m) \rightarrow AC((0, T), \mathbb{R}^k)$, $\gamma = (\gamma_1, \dots, \gamma_m) \mapsto (\gamma_1, \dots, \gamma_k)$. Indeed, Θ^m is obtained as the limit of measures obtained from the flow map of regularizations of V^m . Thus we obtain as their projective limit a probability measure Θ on $AC((0, T), \mathbb{R}^\infty)$, where \mathbb{R}^∞ is equipped with the l_∞ distance. Note that the curves only take values in $Y^\infty = \{x \in \mathbb{R}^\infty : \exists \eta \in \mathcal{P}_*(\mathbb{R}^d) \text{ s.t. } x_i = \langle f_i, \eta \rangle\}$ and that for each $x \in Y^\infty$ there is a unique $\eta \in \mathcal{P}_*(\mathbb{R}^d)$ with $\langle f_i, \eta \rangle = x_i$ for all i . By Kantorovich duality the l_∞ distance corresponds to the W_1 distance. We can thus view Θ as a measure on $AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), W_1))$. Then we check that $(e_t)_\# \Theta = c_t$ for all t . Furthermore, one can check that Θ -almost every curve $(\eta_t)_t$ satisfies in particular for all i

$$\int_0^T a'(t) \int f_i d\eta_t dt = - \int_0^T a(t) \int \bar{\nabla} f_i d\mathcal{U}_{\eta_t, t} dt.$$

In other words, $(\eta_t, \mathcal{U}_{\eta_t, t})$ solves the collision rate equation. Hence $|\dot{\eta}|^2(t) \leq \mathcal{A}(\eta_t, \mathcal{U}_{\eta_t, t})$ and Θ is supported on $AC((0, T), (\mathcal{P}_*(\mathbb{R}^d), \mathcal{W}_B))$. \square

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